ISOTHERMIC SURFACES: CONFORMAL GEOMETRY, CLIFFORD ALGEBRAS AND INTEGRABLE SYSTEMS

F.E. BURSTALL

INTRODUCTION

Manifesto. My aim is to give an account of the theory of isothermic surfaces in $\mathbb{R}^n$ from the point of view of classical surface geometry and also from the perspective of the modern theory of integrable systems and loop groups.

There is some novelty even to the classical theory which arises from the fact that isothermic surfaces are conformally invariant objects in contrast, for example, to the more familiar surfaces of constant Gauss or mean curvature. Thus we have to do with a second order, parabolic geometry which has its own flavour quite unlike Euclidean or Riemannian geometry. To compute effectively in this setting, we shall develop an efficient calculus based on Clifford algebras.

The recent renaissance in interest in isothermic surfaces is principally[1] due to the fact that they constitute an integrable system. I shall attempt to explain in what sense this is true and how this relates to the classical geometry. In particular, I shall show how the loop group formalism provides a context of considerable generality in which results of Bianchi, Darboux and others can be understood and generalised.

All of this will take some preparation so let us begin with an overview of integrable geometry in general and isothermic surfaces in particular.

Background.

What is an integrable system? This is a question with many answers of varying degrees of precision, generality and plausibility! For our present purposes, I take an integrable system to be a geometric object or system of PDE with some (or all) of the following features:

- an infinite-dimensional symmetry group;
- the possibility of writing down explicit solutions;
- a Hamiltonian formulation in which the system is completely integrable in the sense of Liouville.

For Analysts, the prototype example of such a system is the Korteweg–DeVries equation [30] or, perhaps, the non-linear Schrödinger equation [36] but, for Geometers, the basic example, already well-known by the end of the 19th Century, is that of pseudo-spherical surfaces: surfaces in $\mathbb{R}^3$ with constant Gauss curvature $K = -1$.

Let us recall a little of this theory to fix ideas: let $f : M \to \mathbb{R}^3$ be an isometric immersion with $K = -1$. According to Bäcklund, (see [32, §120]), one can solve a

---

[1]Other motivations are available: see, for example, the recent work of Kamberov-Pedit-Pinkall [48] on the Bonnet problem.

first order Frobenius integrable differential equation to obtain a second immersion $\hat{f} : M \to \mathbb{R}^3$, also with $K = -1$ determined by the geometric conditions that

1. $\hat{f} - f$ is of constant length and tangent to both $\hat{f}$ and f;
2. normals at corresponding points of f and $\hat{f}$ make constant angle with each other.

This is the original Bäcklund transformation of pseudo-spherical surfaces. In this procedure there are two parameters: the angle σ between the normals and an initial condition (also an angle) for the differential equation.

Bianchi [32, §121] discovered a beautiful relation between iterated Bäcklund transformations: the permutability theorem. To describe this, we need a little notation: for f a pseudo-spherical surface, let $\mathcal{B}_\sigma f$ denote a Bäcklund transformation of f with angle σ between the normals. Now start with f and let $f_1 = \mathcal{B}_{\sigma_1} f$ and $f_2 = \mathcal{B}_{\sigma_2} f$ be two such Bäcklund transformations. Then Bianchi's theorem asserts the existence of a fourth pseudo-spherical surface $\hat{f}$ which is simultaneously a Bäcklund transformation of f_1 and f_2:

$$\hat{f} = \mathcal{B}_{\sigma_1} f_2 = \mathcal{B}_{\sigma_2} f_1.$$

Moreover $\hat{f}$ can be computed *algebraically* from f, f_1, f_2. In this way, we begin to see the first two of our desiderata for integrability: the Bäcklund transformations generate an infinite-dimensional symmetry group acting on the set of pseudo-spherical surfaces and the permutability theorem shows the possibility of writing down explicit solutions starting with a simple (possibly degenerate) f.

A modern view-point on these classical matters is provided by the theory of loop groups. The group generated by the Bäcklund transformations can be identified as the group of rational maps of the Riemann sphere $\mathbb{P}^1$ into the complex orthogonal group $\mathrm{SO}(3, \mathbb{C})$ satisfying the conditions:

1. $g(0) = 1$ and g is holomorphic at ∞;
2. $g(\lambda) \in \mathrm{SO}(3)$ when $\lambda \in \mathbb{R}$;
3. for all $\lambda \in \mathrm{dom}(g)$, $g(-\lambda) = \tau g(\lambda)$ where τ is a certain involution of $\mathrm{SO}(3, \mathbb{C})$.

In this setting, the generators which act by Bäcklund transformations are distinguished by having a pair of simple poles only[2] while the permutability theorem amounts to an assertion about products of these generators. This view-point is expounded in detail in the recent work of Terng–Uhlenbeck [67] and we shall have much to say about it below.

Where does integrability come from? A starting point from which all this rich structure can be derived is a *zero-curvature formulation* of the underlying problem. That is, the equations describing the problem should amount to the flatness of a family of connections depending on an auxiliary parameter.

Again, we illustrate the basic idea with the example of pseudo-spherical surfaces: a pseudo-spherical surface f admits Chebyshev coordinates ξ, η, that is, asymptotic coordinates for which the coordinate vector fields $\partial/\partial\xi, \partial/\partial\eta$ have unit length. Now let $\theta : M \to \mathbb{R}$ be the angle between these vector fields: the Gauss–Codazzi

---

[2]The position of the poles prescribes the angle σ while the residues there amount to the initial condition of the differential equation.

equations for f amount to a single equation, the *sine-Gordon* equation for θ:

$$\theta_{\xi\eta} = \sin\theta, \tag{0.1}$$

where, here and below, subscripts denote partial differentiation.

Now contemplate the pencil of connections

$$\nabla^\lambda = \mathrm{d} + \begin{pmatrix} 0 & -\theta_\xi & \lambda \\ \theta_\xi & 0 & 0 \\ -\lambda & 0 & 0 \end{pmatrix} \mathrm{d}\xi + \begin{pmatrix} 0 & 0 & \lambda^{-1}\cos\theta \\ 0 & 0 & \lambda^{-1}\sin\theta \\ -\lambda^{-1}\cos\theta & -\lambda^{-1}\sin\theta & 0 \end{pmatrix} \mathrm{d}\eta.$$

By examining the coefficients of λ in the curvature of ∇^λ, it is not difficult to show that ∇^λ is flat for all $\lambda \in \mathbb{R}$ if and only if θ solves (0.1). Thus each pseudo-spherical surface gives rise to a pencil of flat connections.

In fact, more is true: trivialising each ∇^λ produces, at least locally, gauge transformations $F_\lambda : M \to \mathrm{SO}(3)$ intertwining ∇^λ and the trivial connection d and from these one can construct a 1-parameter family of pseudo-spherical surfaces deforming f—these turn out to be the *Lie transforms* of f [32, §122].

These constructions are the starting point of a powerful and rather general method for establishing integrability. Indeed, a zero-curvature formulation of a problem should yield:

- an action of a loop group on solutions;
- a spectral deformation of solutions analogous to the Lie transforms of pseudo-spherical surfaces;
- explicit solutions via Bäcklund transformations or via algebraic geometry.

This theory has been fruitfully applied to a number of geometric problems such as harmonic maps of surfaces into (pseudo-)Riemannian symmetric spaces [9, 10, 46, 70]; isometric immersions of space forms in space forms [39, 65]; flat Egoroff metrics [66]—these include (semisimple) Frobenius manifolds [31, 45] and affine spheres [5].

Isothermic surfaces in $\mathbb{R}^3$. I will describe another classical differential geometric theory that fits into this general picture: this is the theory of *isothermic surfaces*. Classically, a surface in $\mathbb{R}^3$ is isothermic if, away from umbilic points, it admits *conformal curvature line coordinates*, that is, conformal coordinates that, additionally, diagonalise the second fundamental form [18]. Here are some examples:

- surfaces of revolution;
- quadrics;
- minimal surfaces and, more generally, surfaces of constant mean curvature.

There is a second characterisation of isothermic surfaces due to Christoffel [21] : a surface $f : M \to \mathbb{R}^3$ is isothermic if and only if, locally, there is a second surface, a *dual surface*, $f^c : M \to \mathbb{R}^3$ with parallel tangent planes to those of f which induces the same conformal structure but opposite orientation on M. It is this view-point we shall emphasise below.

Isothermic surfaces were studied intensively at the turn of the 20th century and a rich transformation theory of these surfaces was developed that is strikingly reminiscent of that of pseudo-spherical surfaces. Darboux [27] discovered a transformation of isothermic surfaces very like the Bäcklund transformation of pseudo-spherical surfaces: again the transform is effected by solving a Frobenius integrable system of differential equations and again there is a geometric construction only now the surface and its transform are enveloping surfaces of a sphere congruence rather than

focal surfaces of a line congruence. Moreover, Bianchi [2] proved the analogue of his permutability theorem for these Darboux transformations. Again, Bianchi [3], and independently, Calapso [15] found a spectral deformation of isothermic surfaces, the T-transform, strictly analogous to the Lie transform of pseudo-spherical surfaces.

However, there is one important difference between the two theories: that of pseudo-spherical surfaces is a Euclidean theory while that of isothermic surfaces is a conformal one—the image of an isothermic surface by a conformal diffeomorphism of $\mathbb{R}^3 \cup \{\infty\}$ is also isothermic.

Such an intricate transformation theory strongly suggests the presence of an underlying integrable system. That this is indeed the case was established by Cieśliński–Goldstein–Sym [24] who wrote down a zero-curvature formulation of the Gauss–Codazzi equations of an isothermic surface. This work was taken up in [11] where the conformal invariance of the situation was emphasised and the underlying integrable system was identified as an example of the *curved flat* system of Ferus–Pedit [38]. A new view-point on these matters was provided by the Berlin school and their collaborators who developed a beautiful quaternionic formalism for treating surfaces in 4-dimensional conformal geometry [8, 41, 43, 44, 48, 57]. In particular, Hertrich-Jeromin–Pedit [44] discovered a description of Darboux transformations via solutions of a Riccati equation which gives an extraordinarily efficient route into the heart of the theory.

Overview. My purpose in this paper is two-fold: firstly, I want to describe how the entire theory of isothermic surfaces of $\mathbb{R}^3$ can be carried through for isothermic surfaces in $\mathbb{R}^n$ *with no loss of integrable structure.* Secondly, I shall show how this theory can be profitably described using the loop group formalism and, in particular, how to identify Darboux transformations with the dressing action of *simple factors* in the spirit of Terng–Uhlenbeck [66, 67]. In this way, I hope to exhibit the common mechanism underlying the classical geometry of both pseudo-spherical and isothermic surfaces. Along the way, I shall describe a very efficient method for doing conformal geometry which was inspired by the quaternionic formalism of Hertrich-Jeromin–Pedit and, in fact, simultaneously generalises and (at least when $n = 4$) simplifies their approach.

Having declared our aims, let us turn to a more detailed description of the topics we treat. These can be grouped under three headings: isothermic surfaces, loop groups and conformal geometry.

Isothermic surfaces in $\mathbb{R}^n$. An isothermic surface in $\mathbb{R}^n$ can be defined just as in the classical situation: either as a surface admitting conformal curvature line coordinates (although we must now demand that the surface have flat normal bundle in order for curvature lines to be defined) or as a surface that admits a dual surface, that is, a second surface with parallel tangent planes to the first, the same conformal structure and opposite orientation. That these two characterisations locally coincide is due to Palmer [56].

The starting point of our study is the observation that two immersions $f, f^c : M \to \mathbb{R}^n$ are dual isothermic surfaces if and only if

$$(0.2) \qquad\qquad\qquad \mathrm{d}f \wedge \mathrm{d}f^c = 0,$$

where we multiply the coefficients of these $\mathbb{R}^n$-valued 1-forms using the product of the Clifford algebra $C\ell_n$ of $\mathbb{R}^n$. Equation (0.2) is the integrability condition for a

Riccati equation involving an auxiliary parameter $r \in \mathbb{R}^{\times}$:

$$\mathrm{d}g = rg\mathrm{d}f^c g - \mathrm{d}f$$

where again all multiplications take place in $C\ell_n$. We now construct a new isothermic surface $\hat{f}$ by setting $\hat{f} = f + g$: this is the Darboux transform of f. We show that, just as in the classical case, f and $\hat{f}$ are characterised by the conditions that they have the same conformal structure and curvature lines and are the enveloping surfaces of a 2-sphere congruence. This is perhaps, a surprising result: a generic congruence of 2-spheres in $\mathbb{R}^n$ has no enveloping surfaces at all!

This approach to the Darboux transform is a direct extension of that of Hertrich-Jeromin–Pedit for the case $n = 3, 4$ and we follow their methods to prove the Bianchi permutability theorem for Darboux transforms. This proceeds by establishing an explicit algebraic formula for the fourth isothermic surface which comes from the ansatz that corresponding points on the four surfaces in the Bianchi configuration should have constant (Clifford algebra) cross-ratio. We shall find some a priori justification for this ansatz. Further analysis of the Clifford algebra cross-ratio allows us to extend other results of Bianchi in this area to n dimensions: in particular, we prove that the Darboux transform of a Bianchi quadrilateral is another such.

Again, isothermic surfaces in $\mathbb{R}^n$ are conformally invariant and admit a spectral deformation, the T-transform. We explain the intricate relationships between the T-transforms and Darboux transforms of an isothermic surface and its dual.

Examples of isothermic surfaces in $\mathbb{R}^3$ are provided by surfaces of constant mean curvature. In fact, non-minimal CMC surfaces can be characterised as those isothermic surfaces whose dual surface is also a Darboux transform [2] and such surfaces are preserved by a co-dimension 1 family of Darboux transforms. In $\mathbb{R}^n$, we find an exactly analogous theory for *generalised H-surfaces*, that is, surfaces which admit a parallel isoperimetric section in the sense of Chen [20]. In fact, these methods have a wider applicability: applying the same formal arguments in a different algebraic setting establishes the existence of a family of Bäcklund transformations of Willmore surfaces in S^4 [8].

We complete our extension of the classical theory to n dimensions by considering the approach of Calapso [15] who showed that an isothermic surface together with its T-transforms amounts to a solution $\kappa : M \to \mathbb{R}$ of the Calapso equation

$$(0.3) \qquad \Delta\left(\frac{\kappa_{xy}}{\kappa}\right) + 2(\kappa^2)_{xy} = 0.$$

A straightforward generalisation of the analysis in [11] shows that the same is true in $\mathbb{R}^n$ if (0.3) is replaced by a *vector Calapso equation*:

$$\kappa_{xy} = \psi\kappa$$

$$\Delta\psi + 2\left(\sum_{i=1}^{n-2} \kappa_i\right)_{xy} = 0$$

for $\kappa : M \to \mathbb{R}^{n-2}$ and $\psi : M \to \mathbb{R}$. Moreover, we identify κ as (the components of) the conformal Hopf differential of the isothermic surface.

Several of these results have been proved independently by Schief [62] who, in particular, established the existence and Bianchi permutability of Darboux transforms in this context as well as the description of isothermic surfaces via the vector Calapso equation.

Curved flats are submanifolds of a symmetric space on whose tangent spaces the curvature operator vanishes [38]. Curved flats admit a zero-curvature representation and so the methods of integrable systems theory apply. A main result of [11] is that an isothermic surface in $\mathbb{R}^3$ together with a Darboux transform $\hat{f}$ constitute a curved flat $(f, \hat{f}) : M \to S^3 \times S^3 \setminus \Delta$ in the space of pairs of distinct points in $S^3 = \mathbb{R}^3 \cup \{\infty\}$. This last is a pseudo-Riemannian symmetric space for the diagonal action of the Möbius group of conformal diffeomorphisms of S^3. This result goes through unchanged in the n-dimensional setting where we identify the spectral deformation of curved flats with the T-transforms of the factors. In fact, more is true: a curved flat and its spectral deformations give rise via a limiting procedure (Sym's formula) to a certain map of M into a tangent space $\mathfrak{p}$ of the symmetric space. We call these $\mathfrak{p}$-*flat maps* and show that the converse holds: a $\mathfrak{p}$-flat map gives rise to a family of curved flats. In the case of isothermic surfaces, a $\mathfrak{p}$-flat map is the same as an isothermic surface together with a dual surface and our procedure produces the family of the T-transforms of this dual pair. By passing to frames of this family, one obtains an extended object which can be viewed as a map from M into an infinite dimensional group of holomorphic maps $\mathbb{C} \to \varnothing[n+2, \mathbb{C}]$. This is the key to the application of loop group methods to isothermic surfaces to which we now turn.

Loop groups. There is a very general mechanism, pervasive in the theory of integrable systems, for constructing a group action on a space of solutions. Here is the basic idea: let $\mathcal{G}$ be a group with subgroups $\mathcal{G}_1, \mathcal{G}_2$ such that $\mathcal{G}_1 \mathcal{G}_2 = \mathcal{G}$ and $\mathcal{G}_1 \cap \mathcal{G}_2 = \{1\}$. Then $\mathcal{G}_2 \cong \mathcal{G}/\mathcal{G}_1$ so that we get an action of $\mathcal{G}$ and, in particular, $\mathcal{G}_1$ on $\mathcal{G}_2$. In concrete terms, for $g_i \in \mathcal{G}_i$, the product $g_1 g_2$ can be written in a unique way

$$(0.4) \qquad\qquad\qquad g_1 g_2 = \hat{g}_2 \hat{g}_1$$

with $\hat{g}_i \in \mathcal{G}_i$ and then the action is given by

$$g_1 \# g_2 = \hat{g}_2.$$

More generally, when $\mathcal{G}_1 \mathcal{G}_2$ is only open in $\mathcal{G}$, one gets a local action.

The case of importance to us is when the $\mathcal{G}_i$ are groups of holomorphic maps from subsets of the Riemann sphere $\mathbb{P}^1$ to a complex Lie group $G^{\mathbb{C}}$ distinguished by the location of their singularities. For example, in our applications to isothermic surfaces, we take $\mathcal{G}_2$ to be a group of holomorphic maps $\mathbb{C} \to \varnothing[n+2, \mathbb{C}]$ and $\mathcal{G}_1$ a group of rational maps from $\mathbb{P}^1$ to $\varnothing[n+2, \mathbb{C}]$ which are holomorphic near 0 and ∞. The whole point is that, as we have indicated above, isothermic surfaces give rise to certain maps, *extended flat frames*, $M \to \mathcal{G}_2$ of a type that is preserved by the point-wise action of $\mathcal{G}_1$. In this way, we find a local action of $\mathcal{G}_1$ on the set of (dual pairs of) isothermic surfaces and, more generally, on the set of $\mathfrak{p}$-flat maps.

This is a phenomenon that is not peculiar to isothermic surfaces: the key ingredient is that the extended frame is characterised completely by the singularities of its derivative and this ingredient is shared by many integrable systems with a zero-curvature representation (see [67] for many examples).

It remains to compute this action which amounts to performing the factorisation (0.4). In general, this is a Riemann–Hilbert problem for which explicit solutions are not available. However, it is philosophy developed by Terng and Uhlenbeck [66, 67, 70, 71] that there should be certain basic elements of $\mathcal{G}_1$, the *simple factors*, for which the factorisation (0.4) can be computed explicitly and, moreover, the action of these simple factors should amount to Bäcklund-type transformations of

the underlying geometric problem. A difficulty with this approach is that simple factors for a given situation are constructed on an ad hoc basis. We shall propose a concrete characterisation of simple factors which has the status of a theorem when the underlying geometry is that of a compact Riemannian symmetric space and that of an ansatz in non-compact situations (the case of relevance to isothermic surfaces).

For isothermic surfaces, we show that the action of the simple factors we find in this way amount to the Darboux transformations. As a consequence, we find simple complex-analytic arguments that provide a second proof of the circle of results around Bianchi permutability which apply in a variety of contexts.

Conformal geometry and Clifford algebras. The underlying setting for our theory of isothermic surfaces is that of conformal geometry: the basic objects of study are conformally invariant as are many of our ingredients and constructions: sphere congruences, Darboux and T-transforms. This explains the appearance of the indefinite orthogonal group $\mathrm{O}^+(n+1,1)$ and its complexification $\varnothing[n+2,\mathbb{C}]$ as this is precisely the group of conformal diffeomorphisms of $S^n = \mathbb{R}^n \cup \{\infty\}$. Indeed, S^n can be identified with the projective light-cone of $\mathbb{R}^{n+1,1}$ and then the projective action of $\varnothing$ is by conformal diffeomorphisms giving an isomorphism of an open subgroup $\mathrm{O}^+(n+1,1)$ with the Möbius group.

The presence of Clifford algebras in this context, while not new, is not so well known and deserves further comment. Everyone knows how the conformal diffeomorphisms of the Riemann sphere are realised on $\mathbb{C}$ by the action of $\mathrm{SL}(2,\mathbb{C})$ through linear fractional transformations. There is a completely analogous theory in higher dimensions due to Vahlen [72] that replaces $\mathrm{SL}(2,\mathbb{C})$ with a group of 2×2 matrices with entries in a Clifford algebra. Here is the basic idea: instead of working with $\mathrm{O}^+(n+1,1)$, we pass to a double cover and work with an open subgroup of $\mathrm{Pin}(n+1,1)$ which is itself a multiplicative subgroup of the Clifford algebra $C\ell_{n+1,1}$ of $\mathbb{R}^{n+1,1}$. The point now is that $C\ell_{n+1,1}$ is isomorphic to the algebra of 2×2 matrices with entries in the Clifford algebra $C\ell_n$ of $\mathbb{R}^n$. Moreover, a theorem of Vahlen identifies the matrices that comprise the double cover of $\mathrm{O}^+(n+1,1)$ and once again these act on $\mathbb{R}^n$ by linear fractional transformations.

This beautiful formalism is well suited to the study of isothermic surfaces: it makes the action of the Möbius group on $\mathbb{R}^n$ particularly easy to understand and leads to extremely compact formulae in moving frame calculations and elsewhere. While Vahlen's ideas have been used in hyperbolic geometry (see, for example, [34, 35, 73]) and harmonic analysis [40], I believe that this is the first time[3] that these methods have been used in a thorough-going way to do conformal differential geometry.

Road Map. To orient the Reader, we briefly outline the contents of each section of the paper.

Section 1 is preparatory in nature: we describe the light-cone model of the conformal n-sphere and introduce submanifold geometry in this context. We set up the approach via Clifford algebras and use it to prove some preliminary results.

Section 2 contains our account of the classical geometry of isothermic surfaces in $\mathbb{R}^n$. We define Christoffel, Darboux and T-transformations of these surfaces and investigate the permutability relations between them. We consider the special case of generalised H-surfaces and digress to contemplate the vector Calapso equation.

---

[3]See, however, Cieśliński's direct use of $C\ell_{4,1}$ in his study of isothermic surfaces in $\mathbb{R}^3$ [23].

Section 3 is devoted to curved flats. We describe the relation between curved flats and $\mathfrak{p}$-flat maps and how it specialises to give the relation between a dual pair of isothermic surfaces and their T-transforms. We shall see that much of our preceding theory of isothermic surfaces is unified by this curved flats interpretation.

Section 4 deals with loop groups and how they may be applied to study curved flats in general and isothermic surfaces in particular. We give a general discussion of simple factors and then specialise to give a detailed account of the case of isothermic surfaces.

Section 5 rounds things off with brief descriptions of recent developments and some open problems.

A note on the text. This work had its genesis in lecture notes for a short course on "Integrable systems in conformal geometry" given at Tsing Hua University in January 1999 but has evolved into a statement of Everything I Know About Isothermic Surfaces. However, this final version has retained something of its origins in that I have given a somewhat leisurely account of background material and also in that I have set a large number of exercises. These exercises are an integral part of the exposition and, among other things, contain most of the computations where no New Idea is needed. Solutions may become available at

> http://www.maths.bath.ac.uk/~feb/taiwan-solutions.html

and Readers are warmly invited to contribute their own!

Note added in October 2000. Some time after this paper was written, I have had the opportunity to read a wonderful book by Tzitzéica [69] which contains a completely different approach to isothermic surfaces: Tzitzéica studies surfaces in an n-dimensional projective quadric that support a conjugate net with equal Laplace invariants. He observes that, when $n = 3$, these are exactly the isothermic surfaces (the conjugate net is that formed by curvature lines while the quadric is the projective light-cone of our exposition) and develops a theory of Darboux transformations of such surfaces for arbitrary n. It is not hard to see that, for any n, Tzitzéica's

---

surfaces amount (locally) to precisely the isothermic surfaces in the conformal compactification of some $\mathbb{R}^{p,q}$ with $p + q = n$. Thus, in this way, isothermic surfaces and their Darboux transformations have been known in this generality since 1924!

1. Conformal geometry and Clifford algebras

1.1. **Conformal geometry of S^n.** Recall that a map $\phi : (M, g) \to (M, g)$ of a Riemannian manifold is *conformal* if $\mathrm{d}\phi$ preserves angles. Analytically this means

$$\phi^* g = e^{2u} g$$

for some $u : M \to \mathbb{R}$.

Here are some conformal maps of (open sets of) $\mathbb{R}^n$:

1. Euclidean motions: $\phi^* g = g$;
2. Dilations: $x \mapsto rx$, $r \in \mathbb{R}^+$;
3. Inversions in hyperspheres: for fixed $p \in \mathbb{R}^n$, $r \in \mathbb{R}^+$, these are $\phi : \mathbb{R}^n \setminus \{p\} \to \mathbb{R}^n$ given by

$$\phi(x) = p + r^2 \frac{x - p}{\|x - p\|^2}.$$

$$\|\phi(x) - p\| = \frac{r^2}{\|x - p\|}$$
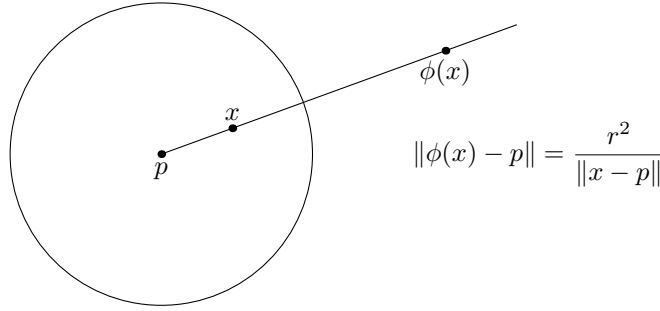
Figure 1. Inversion in the sphere of radius r about p

Exercise 1.1. Show that such inversions are conformal.

A theorem of Liouville states:

Theorem 1.1. *For $n \geq 3$, any conformal map $\Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ is the restriction to Ω of a composition of Euclidean motions, dilations and inversions.*

For a proof, see do Carmo [29].

It is natural to extend the definition of the inversions $\phi : \mathbb{R}^n \setminus \{p\} \to \mathbb{R}^n$ by setting $\phi(p) = \infty$ and $\phi(\infty) = p$ and so viewing ϕ as a conformal diffeomorphism of the n-sphere $\mathbb{R}^n \cup \{\infty\} = S^n$. To make sense of this, recall that the conformal geometries of $\mathbb{R}^n$ and $S^n \subset \mathbb{R}^{n+1}$ are linked by stereographic projection: choosing a "point at infinity" $v_\infty \in S^n$, we have a conformal diffeomorphism $\pi : S^n \setminus \{v_\infty\} \to \langle v_\infty \rangle^\perp \cong \mathbb{R}^n$ as in Figure 2.

Exercise 1.2. Prove:

1. π is a conformal diffeomorphism;
2. $S \subset \mathbb{R}^n$ is a k-sphere if and only if $\pi^{-1}(S) \subset S^n$ is a k-sphere;

FIGURE 2. Stereographic projection

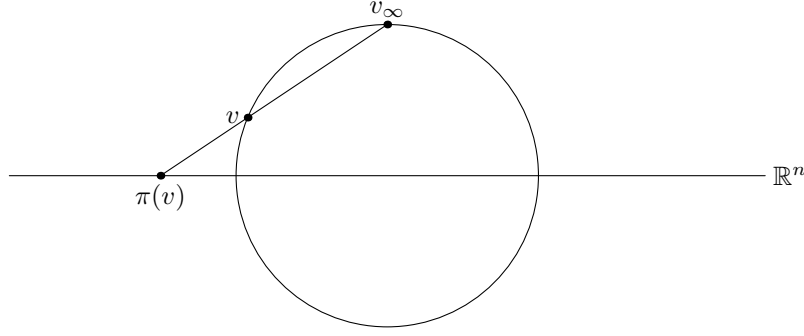3. $V \subset \mathbb{R}^n$ is an affine k-plane if and only if $\pi^{-1}(V) \cup \{v_\infty\} \subset S^n$ is a k-sphere containing v_∞. Thus "planes are spheres through infinity".

Under stereographic projection, inversions in hyperspheres extend to conformal diffeomorphisms of S^n as do Euclidean motions and dilations (these fix v_∞) and, in this way, we are led to consider the *Möbius group* Möb(n) of conformal diffeomorphisms of S^n.

To go further, it is very convenient to introduce another model of the n-sphere discovered by Darboux[5] [26]. For this, we contemplate the Lorentzian space $\mathbb{R}^{n+1,1}$: a real $(n+2)$-dimensional vector space equipped with an inner product $(\,,\,)$ of signature $(n+1,1)$ so that there is an orthonormal basis $e_1, \ldots, e_{n+2}$ with

$$(e_i, e_i) = \begin{cases} 1 & i < n+2 \\ -1 & i = n+2. \end{cases}$$

Inside $\mathbb{R}^{n+1,1}$, we distinguish the *light-cone* L:

$$L = \{v \in \mathbb{R}^{n+1,1} \setminus \{0\} : (v,v) = 0\}.$$

Exercise 1.3. L is a submanifold of $\mathbb{R}^{n+1,1}$.

Clearly, if $v \in L$ and $r \in \mathbb{R}^\times$ then $rv \in L$ so that $\mathbb{R}^\times$ acts freely on L and we may take the quotient $\mathbb{P}(L) \subset \mathbb{P}(\mathbb{R}^{n+1,1})$:

$$\mathbb{P}(L) = L/\mathbb{R}^\times = \{\ell \subset \mathbb{R}^{n+1,1} : \ell \text{ is a 1-dimensional isotropic subspace}\}.$$

The point of this is that $\mathbb{P}(L)$ has a conformal structure with respect to which it is conformally diffeomorphic to S^n with its round metric. Indeed, let us fix a unit time-like vector $t_0 \in \mathbb{R}^{n+1,1}$ (thus $(t_0, t_0) = -1$) and set

$$S_{t_0} = \{v \in L : (v, t_0) = -1\}.$$

For $v \in S_{t_0}$, write $v = v^\perp + t_0$ so that $v^\perp \perp t_0$ and note:

$$0 = (v,v) = (v^\perp, v^\perp) + (t_0, t_0) = (v^\perp, v^\perp) - 1.$$

Thus the projection $v \mapsto v^\perp$ is a diffeomorphism $S_{t_0} \to S^n$ onto the unit sphere in $\langle t_0 \rangle^\perp \cong \mathbb{R}^{n+1}$ which is easily checked to be an isometry.

Exercise 1.4.   1. For $v \in L$, $(t_0, v) \neq 0$.
    2. Deduce that each line $\ell \in \mathbb{P}(L)$ intersects S_{t_0} in exactly one point.

---

[5]For a modern account, see Bryant [7].

Thus we have a diffeomorphism $\ell \mapsto S_{t_0} \cap \ell : \mathbb{P}(\mathrm{L}) \to S_{t_0}$ whose inverse is the canonical projection $\pi : v \mapsto \langle v \rangle : \mathrm{L} \to \mathbb{P}(\mathrm{L})$ restricted to S_{t_0}.

Exercise 1.5. Suppose that t_0' is another unit time-like vector. Show that the composition $S_{t_0'} \xrightarrow{\pi} \mathbb{P}(\mathrm{L}) \cong S_{t_0}$ is a conformal diffeomorphism which is *not* an isometry unless $t_0' = \pm t_0$.

To summarise the situation: *for each unit time-like t_0, π restricts to a diffeomorphism $S_{t_0} \to \mathbb{P}(\mathrm{L})$ and each such diffeomorphism induces a conformally equivalent metric on $\mathbb{P}(\mathrm{L})$.*

Exercise 1.6. Let g be any Riemannian metric on S^n in the conformal class of the round metric. Show that there is an isometric embedding $(S^n, g) \to \mathrm{L}$.

Having identified $\mathbb{P}(\mathrm{L})$ with the conformal n-sphere, we can use a similar argument to describe stereographic projection in this model by replacing time-like t_0 with $v_\infty \in \mathrm{L}$: fix $v_0, v_\infty \in \mathrm{L}$ with $(v_0, v_\infty) = -\frac{1}{2}$ (so that, in particular, $\langle v_0 \rangle \neq \langle v_\infty \rangle$) and set

$$E_{v_\infty} = \{v \in \mathrm{L} : (v, v_\infty) = -\tfrac{1}{2}\}.$$

Exercise 1.7. For $v \in \mathrm{L} \setminus \langle v_\infty \rangle$, show that $(v, v_\infty) \neq 0$.

Thus we have a diffeomorphism $\ell \mapsto E_{v_\infty} \cap \ell : \mathbb{P}(\mathrm{L}) \setminus \{\langle v_\infty \rangle\} \to E_{v_\infty}$. Moreover, E_{v_∞} is isometric to a Euclidean space: indeed, set $\mathbb{R}^n = \langle v_0, v_\infty \rangle^\perp$, a subspace of $\mathbb{R}^{n+1,1}$ on which the inner product is definite.

Exercise 1.8.   1. There is an isometry $E_{v_\infty} \to \mathbb{R}^n$ given by

(1.1) $$v \mapsto v - v_0 + 2(v, v_0)v_\infty$$

with inverse

(1.2) $$x \mapsto x + v_0 + (x, x)v_\infty.$$

   2. Verify that the composition $\mathbb{P}(\mathrm{L}) \setminus \langle v_\infty \rangle \cong E_{v_\infty} \to \mathbb{R}^n$ really is stereographic projection.
   More precisely, set $t_0 = v_0 + v_\infty$, $x_0 = v_0 - v_\infty$ so that $(t_0, t_0) = -1 = -(x_0, x_0)$. Let S^n be the unit sphere in $\langle t_0 \rangle^\perp$. Then the composition

   $$S^n \setminus \{x_0\} \to S_{t_0} \setminus \{2v_\infty\} \xrightarrow{\pi} \mathbb{P}(\mathrm{L}) \setminus \langle v_\infty \rangle \to \mathbb{R}^n = \langle v_0, v_\infty \rangle^\perp = \langle t_0, x_0 \rangle^\perp$$

   is stereographic projection.

The beauty of this model is that it *linearises* conformal geometry. For example, observe that the set of hyperspheres in S^n is parametrised by the set $\mathbb{P}^+(\mathbb{R}^{n+1,1})$ of space-like lines, that is, 1-dimensional subspaces on which the inner product is positive definite. Indeed, if $L \subset \mathbb{R}^{n+1,1}$ is such a line then $L^\perp \cong \mathbb{R}^{n,1}$ so that $\mathbb{P}(\mathrm{L} \cap L^\perp) \cong S^{n-1}$. Choosing v_0, v_∞ and so a choice of stereographic projection, this correspondence becomes quite explicit:

Exercise 1.9. Let $L \in \mathbb{P}^+(\mathbb{R}^{n+1,1})$ and fix $s \in L$ of unit length. Let $s^\perp$ be the orthoprojection of s onto $\mathbb{R}^n = \langle v_0, v_\infty \rangle^\perp$ and set $S_L = \mathbb{P}(\mathrm{L} \cap L^\perp)$.

   1. If $\langle v_\infty \rangle \in S_L$, that is, $(v_\infty, s) = 0$, then the stereo-projection of $S_L \setminus \{\langle v_\infty \rangle\}$ is the hyperplane

   $$\{x \in \mathbb{R}^n : (x, s^\perp) = -(s, v_0)\}.$$

   2. If $\langle v_\infty \rangle \notin S_L$, then the stereo-projection of S_L is the sphere centred at $-s^\perp/2(s, v_\infty)$ of radius $1/2|(s, v_\infty)|$.

3. Stereo-projection intertwines reflection in the hyperplane $L^\perp \subset \mathbb{R}^{n+1,1}$ with reflection or inversion in the plane or sphere determined by S_L.

Now contemplate the orthogonal group $\emptyset$ of $\mathbb{R}^{n+1,1}$, that is,

$$\emptyset = \{T \in \mathrm{GL}(n+2, \mathbb{R}) : (Tu, Tv) = (u, v) \text{ for all } u, v \in \mathbb{R}^{n+1,1}\}.$$

The linear action of $\emptyset$ on $\mathbb{R}^{n+1,1}$ preserves $\mathbb{L}$ and the set of lines in $\mathbb{L}$ and so descends to an action on $\mathbb{P}(\mathbb{L})$. Moreover, for t_0 a unit time-like vector and $T \in \emptyset$, T restricts to give an isometry $S_{t_0} \to S_{Tt_0}$ so that the induced map on $\mathbb{P}(\mathbb{L})$ is a *conformal* diffeomorphism. In this way, we have found a homomorphism $\emptyset \to \mathrm{M\ddot{o}b}(n)$ which is, in fact, a double cover:

Theorem 1.2. *The sequence* $0 \to \mathbb{Z}_2 \to \emptyset \to \mathrm{M\ddot{o}b}(n) \to 0$ *is exact.*

Proof. Any T in the kernel of our homomorphism must preserve each light-line and so has each light-line as an eigenspace. This forces T to be a multiple of the identity matrix I and then $T \in \emptyset$ gives $T = \pm I$.

Thus the main issue is to see that our homomorphism is onto. However, by Liouville's Theorem, $\mathrm{M\ddot{o}b}(n)$ is generated by reflections in hyperplanes and inversions in hyperspheres: indeed, any Euclidean motion is a composition of reflections while a dilation is a composition of two inversions in concentric spheres. On the other hand, we have seen in Exercise 1.9 that all these reflections and inversions are induced by reflections in $L^\perp$ for $L \in \mathbb{P}^+(\mathbb{R}^{n+1,1})$ a space-like line. Such reflections are certainly in $\emptyset$ and we are done. $\square$

In fact, we can do better: the light cone $\mathbb{L}$ has two components[6] $\mathbb{L}^+$ and $\mathbb{L}^-$ which are transposed by $v \mapsto -v$. Correspondingly, $\emptyset$ has four components distinguished by the sign of the determinant and whether or not the components of $\mathbb{L}$ are preserved. Denote by $\mathrm{O}^+(n+1, 1)$ the subgroup of $\emptyset$ that preserves $\mathbb{L}^\pm$. Then $-I \notin \mathrm{O}^+(n+1, 1)$ and we deduce:

Theorem 1.3. $\mathrm{O}^+(n+1, 1) \cong \mathrm{M\ddot{o}b}(n)$.

The two components of $\mathrm{O}^+(n+1, 1)$ are the orientation preserving and orientation reversing conformal diffeomorphisms of $\mathbb{P}(\mathbb{L})$.

1.2. Submanifold geometry in $\mathbb{P}(\mathbb{L})$.

1.2.1. Submanifolds and normal bundles. Contemplate the projection $\pi : \mathbb{L} \to \mathbb{P}(\mathbb{L})$, $\pi(v) = \langle v \rangle$. Clearly, $T_v \mathbb{L} = \langle v \rangle^\perp$ while $\ker \mathrm{d}\pi_v = \langle v \rangle$ so we have an isomorphism

$$\mathrm{d}\pi_v : \langle v \rangle^\perp / \langle v \rangle \cong T_{\langle v \rangle} \mathbb{P}(\mathbb{L}).$$

Scaling v leaves $\langle v \rangle^\perp / \langle v \rangle$ unchanged but scales the isomorphism:

Exercise 1.10. For $r \in \mathbb{R}^\times$ and $X \in \langle v \rangle^\perp / \langle v \rangle$, $\mathrm{d}\pi_{rv}(rX) = \mathrm{d}\pi_v(X)$.

More invariantly, we have an isomorphism $\mathrm{Hom}(\langle v \rangle, \langle v \rangle^\perp / \langle v \rangle) \cong T_{\langle v \rangle} \mathbb{P}(\mathbb{L})$ given by

$$B \mapsto \mathrm{d}\pi_v(Bv)$$

which is well-defined by Exercise 1.10.

For $\ell \in \mathbb{P}(\mathbb{L})$, the inner product on $\mathbb{R}^{n+1,1}$ induces a positive definite inner product on $\ell^\perp / \ell$ and if $v \in \ell^\times$ lies in some round sphere S_{t_0} then projection along ℓ is an

---

[6]Non-collinear elements $v_0, v_\infty \in \mathbb{L}$ are in the same component if and only if $(v_0, v_\infty) < 0$.

isometry $T_v S_{t_0} \to \ell^\perp/\ell$. We therefore conclude that the isomorphism $\mathrm{d}\pi_v : \ell^\perp/\ell \to T_\ell \mathbb{P}(\mathrm{L})$ is conformal.

Now let M be a manifold and $\phi : M \to \mathbb{P}(\mathrm{L})$ an immersion. We study ϕ by studying its *lifts*, that is, maps $f : M \to \mathrm{L}$ with $\pi \circ f = \phi$. Since the principal $\mathbb{R}^\times$-bundle $\pi : \mathrm{L} \to \mathbb{P}(\mathrm{L})$ is trivial (each S_{t_0} is the image of a section!) there are many such lifts. Moreover, in view of Theorem 1.3, it suffices to consider lifts $f : M \to \mathrm{L}^+$. If f is one such, then any other is of the form $e^u f$ for some $u : M \to \mathbb{R}$.

So let $f : M \to \mathrm{L}^+$ be a lift of $\phi = \langle f \rangle$. Then $\mathrm{d}\pi_f$ gives an isomorphism

$$\langle f \rangle^{-1} T\mathbb{P}(\mathrm{L}) \cong \langle f \rangle^\perp / \langle f \rangle$$

under which the derivative of $\langle f \rangle$ is given by $\mathrm{d}f \mod \langle f \rangle$. In particular, $\langle f \rangle$ is an immersion if and only if, for each $X \in TM$,

$$\mathrm{d}_X f \wedge f \neq 0.$$

Scaling the lift scales the isomorphism:

$$\mathrm{d}(e^u f) = e^u(\mathrm{d}u f + \mathrm{d}f) \equiv e^u \mathrm{d}f \mod \langle f \rangle$$

from which we see that the image of $\mathrm{d}f$ in $\langle f \rangle^\perp / \langle f \rangle$ is independent of the choice of lift. Thus, when $\langle f \rangle$ is an immersion, orthogonal decomposition gives a well-defined *weightless normal bundle*[7] $\mathcal{N}_{\langle f \rangle}$:

$$(1.3) \qquad \langle f \rangle^\perp / \langle f \rangle = \mathrm{Im}\, \mathrm{d}f(TM) \oplus \mathcal{N}_{\langle f \rangle}$$

which is Möbius invariant: for $T \in \varnothing$,

$$\mathcal{N}_{T\langle f \rangle} = T\mathcal{N}_{\langle f \rangle}.$$

Notation. For s a section of $\langle f \rangle^\perp$, write

$$s + \langle f \rangle = [s]^T + [s]^\perp$$

according to the decomposition (1.3).

1.2.2. *Conformal invariants.* We construct conformal invariants of submanifolds by finding $\mathrm{O}^+(n+1, 1)$-invariant properties of lifts that do not depend on the choice of said lift.

Firstly, we have the conformal class of the metric induced by f on M:

$$\big(\mathrm{d}(e^u f), \mathrm{d}(e^u f)\big) = e^{2u}\big(\mathrm{d}f, \mathrm{d}f\big)$$

since $(\mathrm{d}f, f) = 0$ ($\mathrm{d}f$ is $\langle f \rangle^\perp$-valued).

Now let N be a section of $\langle f \rangle^\perp$ such that $[N] = N + \langle f \rangle$ is a section $\mathcal{N}_{\langle f \rangle}$. Thus

$$(N, f) = (N, \mathrm{d}f) = 0$$

whence

$$(\mathrm{d}N, f) = -(N, \mathrm{d}f) = 0$$

so that $\mathrm{d}N$ is $\langle f \rangle^\perp$-valued also. Moreover, if $[N] = [N']$ so that $N' = N + \mu f$, for some function $\mu : M \to \mathbb{R}$, we have

$$\mathrm{d}N' = \mathrm{d}N + \mu \mathrm{d}f + \mathrm{d}\mu f$$

whence

$$\mathrm{d}N' \equiv \mathrm{d}N + \mu \mathrm{d}f \mod \langle f \rangle.$$

---

[7]Strictly speaking, the normal bundle to $\langle f \rangle$ is $\mathrm{Hom}(\langle f \rangle, \mathcal{N}_{\langle f \rangle}) = \langle f \rangle^* \otimes \mathcal{N}_{\langle f \rangle} \subset \langle f \rangle^{-1} T\mathbb{P}(\mathrm{L})$. Since we will mostly deal with lifts f of $\langle f \rangle$, we shall ignore this distinction which, in any case, amounts only to tensoring with a trivial line bundle.

In particular,

$$[\mathrm{d}N]^\perp = [\mathrm{d}N']^\perp$$

so that we can define a conformally invariant connection $\nabla^\perp$ on $\mathcal{N}_{\langle f \rangle}$ by

$$\nabla^\perp[N] = [\mathrm{d}N]^\perp.$$

Further,

$$\mathrm{d}f^{-1}([\mathrm{d}N']^T) = \mathrm{d}f^{-1}([\mathrm{d}N]^T) + \mu$$

so that shape operators are well-defined up to addition of multiples of the identity and scaling (as the lift varies). In particular, the eigenspaces of shape operators, *the principal curvature directions*, are well-defined.

To summarise: given an immersion $\langle f \rangle : M \to \mathbb{P}(\mathrm{L})$, we obtain in a Möbius invariant way:

1. A conformal class of metrics on M;
2. A weightless normal bundle $\mathcal{N}_{\langle f \rangle}$ with normal connection $\nabla^\perp$;
3. The conformal class of *trace-free* shape operators.

Remark. A more precise formulation of these invariants can be obtained by viewing $\langle f \rangle$ as the sub-bundle of the trivial bundle $M \times \mathbb{R}^{n+1,1}$ whose fibre at $p \in M$ is $\langle f(p) \rangle \subset \mathbb{R}^{n+1,1}$. Following Calderbank [17], our conformal class of metrics on M can be viewed as an honest metric on $TM \otimes \langle f \rangle$ via

$$(X \otimes f, Y \otimes f) = (\mathrm{d}_X f, \mathrm{d}_Y f),$$

$X, Y \in TM$. In the same way, the conformal class of trace-free shape operators can be viewed as a single trace-free quadratic form taking values in $\mathcal{N}_{\langle f \rangle} \otimes \langle f \rangle$. We shall return to this viewpoint on conformal submanifold geometry elsewhere.

When M is an orientable surface, our analysis can be refined somewhat. In this case, M becomes a Riemann surface so let $z = x + iy$ be a holomorphic coordinate and take a lift $f : M \to \mathrm{L}^+$. We define a local section $K_{\langle f \rangle}$ of $\mathcal{N}^*_{\langle f \rangle}$ by

$$K_{\langle f \rangle}(N + \langle f \rangle) = \sqrt{2}\frac{(f_{zz}, N)}{\sqrt{(f_z, f_{\bar{z}})}}$$

where, here and below, we use subscripts to denote partial differentiation.

Exercise 1.11. $K_{\langle f \rangle}$ is well-defined and independent of the choice of lift f in L^+.

It is clear that $K_{\langle f \rangle}$ is equivariant under the action of $\mathrm{O}^+(n + 1, 1)$: for $T \in \mathrm{O}^+(n + 1, 1)$,

$$K_{T\langle f \rangle} \circ T = K_{\langle f \rangle}$$

and so is conformally invariant. As for the dependence on the holomorphic coordinate z, we see that $K_{\langle f \rangle}$ should be viewed a density with values in $\mathcal{N}^*_{\langle f \rangle}$, that is, as a section of $(\bigwedge^{1,0} M)^{3/2} \otimes (\bigwedge^{0,1} M)^{-1/2} \otimes \mathcal{N}^*_{\langle f \rangle}$.

We call $K_{\langle f \rangle}$ the *conformal Hopf differential* of $\langle f \rangle$ and will return to this topic in Section 2.5.

1.2.3. *Spheres and sphere congruences.* We have already seen that the hyperspheres in S^n are parametrised by the space $\mathbb{P}^+(\mathbb{R}^{n+1,1})$ of space-like lines. In the same way, the Grassmannian $G_k^+(\mathbb{R}^{n+1,1})$ of space-like k-planes in $\mathbb{R}^{n+1,1}$ parametrises co-dimension k spheres in S^n [61]. Indeed, any such sphere is of the form $S_\Pi = \mathbb{P}(\Pi^\perp \cap \mathrm{L})$ for a unique $\Pi \in G_k^+(\mathbb{R}^{n+1,1})$.

Now let $\langle f \rangle : M \to \mathbb{P}(\mathrm{L})$ be an immersion and $\Pi \in G_k^+(\mathbb{R}^{n+1,1})$. For $p \in M$, we see that $\langle f \rangle(p) \in S_\Pi$ if and only if $\Pi \perp f(p)$ while $\langle f \rangle$ is tangent to S_Π at p if, in addition, $\Pi \perp \mathrm{Im}\, \mathrm{d}f_p$.

Definition. A *congruence of k-spheres* is a map $\Pi : M \to G_{n-k}^+(\mathbb{R}^{n+1,1})$ of a k-dimensional manifold into the space of k-spheres.

An immersion $\langle f \rangle : M \to \mathbb{P}(\mathrm{L})$ *envelopes* the congruence Π if, for each $p \in M$, the sphere $S_{\Pi(p)}$ has first order contact with $\langle f \rangle$ at p. This amounts to demanding that

(1.4a) $(\Pi, f) = 0$

(1.4b) $(\Pi, \mathrm{d}f) = 0.$

It can be shown that under mild (open) conditions, a congruence of hyperspheres has two enveloping hypersurfaces. In higher co-dimension, there need not be *any* enveloping submanifolds.

In view of (1.4), there is a close relationship between the normal bundle $\mathcal{N}_{\langle f \rangle}$ of an immersion $\langle f \rangle$ and the sphere congruences that envelope $\langle f \rangle$. Indeed, if $\Pi : M \to G_{n-k}^+$ is such a congruence, then (1.4a) says that $\Pi \subset \langle f \rangle^\perp$ and then (1.4b) shows that projection along $\langle f \rangle$ is a isomorphism $\Pi \cong \mathcal{N}_{\langle f \rangle}$. Moreover, this isomorphism is parallel with respect to the connection on Π induced by flat differentiation in $\mathbb{R}^{n+1,1}$ and $\nabla^\perp$ on $\mathcal{N}_{\langle f \rangle}$. In particular, the honest normal bundle of a lift f lying in some Riemannian model of $\mathbb{P}(\mathrm{L})$ gives an enveloping sphere congruence.

Example. Let $f : M \to E_{v_\infty} \subset \mathrm{L}^+$ be a lift lying in a copy of Euclidean space and let $\Pi = \mathrm{d}f(TM)^\perp \subset f^{-1}TE_{v_\infty} = \langle f, v_\infty \rangle^\perp$. Then Π is an enveloping sphere congruence and since $(\Pi, v_\infty) = 0$ we see that each sphere $S_{\Pi(p)}$ meets the point at infinity $\langle v_\infty \rangle$. Thus, after stereo-projection, $S_{\Pi(p)}$ is a plane and Π is the congruence $p \mapsto \mathrm{d}f(T_pM)$ of tangent planes to f.

For a more substantial example, let $f : M \to \mathrm{L}^+$ be a lift of an immersion of a k-dimensional manifold and let $\mathbf{H}_f$ be the mean curvature vector of f:

$$\mathbf{H}_f = \frac{1}{k}\, \mathrm{trace}\, \nabla \mathrm{d}f$$

where ∇ is the connection on $TM \otimes f^{-1}T\mathbb{R}^{n+1,1}$ induced by flat differentiation on $\mathbb{R}^{n+1,1}$ and the Levi–Civita connection for the metric $(\mathrm{d}f, \mathrm{d}f)$ on M (the trace is, of course, computed with respect to this metric also).

Exercise 1.12.   1. The sub-bundle $Z_{\langle f \rangle} = \langle f, \mathrm{d}f, \mathbf{H}_f \rangle^\perp \subset \mathbb{R}^{n+1,1}$ depends only on $\langle f \rangle$ and not on the choice of lift.
    2. For $T \in \emptyset$, $TZ_{\langle f \rangle} = Z_{T\langle f \rangle}$.

Moreover, for $e_1, \ldots, e_k$ orthonormal with respect to $(\mathrm{d}f, \mathrm{d}f)$, we have

$$(f, \mathbf{H}_f) = -\frac{1}{k}(\mathrm{d}_{e_i}f, \mathrm{d}_{e_i}f) \neq 0$$

whence, at each point $\langle f, \mathrm{d}f, \mathbf{H}_f \rangle$ spans a $(k+2)$-plane on which the inner product has signature $(k+1, 1)$ so that each $Z_{\langle f \rangle}(p)$ is a space-like $(n-k)$-plane and $Z_{\langle f \rangle} : M \to G_{n-k}^+(\mathbb{R}^{n+1,1})$ is a Möbius invariant enveloping sphere congruence. Geometrically, for a Euclidean lift $f : M \to E_{v_\infty}$, this is the sphere congruence for which the sphere tangent to f at p has the same mean curvature vector at p as f.

$Z_{\langle f \rangle}$ is the *central sphere congruence* [68] or *conformal Gauss map* [7] of $\langle f \rangle$.

This construction comes alive when $k = 2$ where it becomes a fundamental tool in the theory of Willmore surfaces [7, 33]. We shall meet this congruence again when we discuss Calapso's approach [15, 16] to isothermic surfaces in Section 2.5.

1.3. **Clifford algebras in conformal geometry.** We are going to develop an extraordinarily efficient calculus for conformal geometry using Clifford algebras that is especially well adapted to working in the familiar Euclidean setting. This will take a little preparation so we begin by summarising the main idea.

We already know that the orthogonal group $O^+(n + 1, 1)$ is isomorphic to the Möbius group Möb(n). We shall take a double cover and work instead with an open subgroup of Pin($n + 1, 1$) which lies in the Clifford algebra $C\ell_{n+1,1}$ of $\mathbb{R}^{n+1,1}$. A priori, it is not so clear why this is a useful strategy. However, there is a simple isomorphism of algebras between $C\ell_{n+1,1}$ and the algebra $C\ell_n(2)$ of 2×2 matrices in the Clifford algebra $C\ell_n$ of $\mathbb{R}^n$. The image under this isomorphism of Pin($n + 1, 1$) is identified by a theorem of Vahlen [72]. Using this model, conformal diffeomorphisms of $\mathbb{R}^n$ become linear fractional transformations and the method of the moving frame simplifies massively as one only has to do with 2×2 matrices rather than the $(n + 2) \times (n + 2)$ matrices of the Ø formulation.

A good general reference for Clifford algebras is the text of Michelsohn–Lawson [53, Chapter 1] while a clear account of the relation between Clifford algebras and Möbius transformations can be found in the monograph of Porteous[8] [58, Chapters 18 and 23].

1.3.1. *Clifford algebras.* Let $\mathbb{R}^{p,q}$ denote a $(p+q)$-dimensional vector space equipped with an inner product of signature (p, q) (that is, p positive directions and q negative ones) and let $C\ell_{p,q}$ denote its Clifford algebra. Thus $C\ell_{p,q}$ is an associative algebra with unit 1 of dimension 2^{p+q} which contains $\mathbb{R}^{p,q}$ and is generated by $\mathbb{R}^{p,q}$ subject only to the relations

$$vw + wv = -2(v, w)1.$$

$C\ell_{p,q}$ has a universal property which ensures the existence of the following (anti-) involutions uniquely determined by their action on the generators $\mathbb{R}^{p,q}$:

1. $a \mapsto \tilde{a}$: the *order involution* with $\tilde{v} = -v$ for $v \in \mathbb{R}^{p,q}$.
2. $a \mapsto a^t$: the *transpose anti-involution* with $v^t = v$ for $v \in \mathbb{R}^{p,q}$.
3. $a \mapsto \bar{a}$: the *conjugate anti-involution* with $\bar{v} = -v$ for $v \in \mathbb{R}^{p,q}$.

Exercise 1.13. For $a \in C\ell_{p,q}$, $\tilde{a} = \bar{a}^t$.

The invertible elements $C\ell_{p,q}^\times$ form a multiplicative group which acts on $C\ell_{p,q}$ via the *twisted adjoint action*:

$$\widetilde{\mathrm{Ad}}(g)a = ga\tilde{g}^{-1}$$

Exercise 1.14. $\widetilde{\mathrm{Ad}} : C\ell_{p,q}^\times \to \mathrm{GL}(C\ell_{p,q})$ is a representation.

Inside $C\ell_{p,q}^\times$ we distinguish the *Clifford group* $\Gamma_{p,q}$ given by

$$\Gamma_{p,q} = \{g \in C\ell_{p,q}^\times : \widetilde{\mathrm{Ad}}(g)\mathbb{R}^{p,q} \subset \mathbb{R}^{p,q}\}$$

The twisted adjoint action therefore restricts to give a representation of $\Gamma_{p,q}$ on $\mathbb{R}^{p,q}$.

---

[8]In fact, our approach differs slightly in the details from that in [58] since our conformal diffeomorphisms act on vectors (the $\mathbb{R}^n$ that generates $C\ell_n$) rather than *hypervectors* (spanned by 1 and some $\mathbb{R}^{n-1} \subset \mathbb{R}^n$). In this we have followed [73].

Fact. $\Gamma_{p,q}$ *is generated by* $\mathbb{R}^{p,q}_{\times} = \{v \in \mathbb{R}^{p,q} : (v,v) \neq 0\} = \mathbb{R}^{p,q} \cap C\ell^{\times}_{p,q}$.

Exercise 1.15. For $v \in \mathbb{R}^{p,q}_{\times}$, $\widetilde{\mathrm{Ad}}(v) : w \mapsto vw\tilde{v}^{-1} = -vwv^{-1}$ is reflection in the hyperplane orthogonal to v.

As a consequence, each $\widetilde{\mathrm{Ad}}(g) \in \varnothing[p,q]$, the orthogonal group of $\mathbb{R}^{p,q}$, and $\widetilde{\mathrm{Ad}} : \Gamma_{p,q} \to \varnothing[p,q]$ is a homomorphism which has all reflections in its image and so is surjective by the Cartan–Dieudonné theorem. Moreover, $\ker \widetilde{\mathrm{Ad}} = \mathbb{R}^{\times}$ so that we have an exact sequence:

$$0 \to \mathbb{R}^{\times} \to \Gamma_{p,q} \overset{\widetilde{\mathrm{Ad}}}{\to} \varnothing[p,q] \to 0.$$

For $g \in \Gamma_{p,q}$, set $N(g) = g\bar{g}$, the *norm* of g. Writing $g = v_1 \dots v_n$, with each $v_i \in \mathbb{R}^{p,q}_{\times}$, we see that

$$N(g) = g\bar{g} = (v_1 \dots v_n)\overline{(v_1 \dots v_n)} = (v_1 \dots v_n)(\bar{v}_n \dots \bar{v}_1)$$
$$= \prod_{i=1}^{n}(v_i, v_i) \in \mathbb{R}^{\times}$$

since $v_i\bar{v}_i = -v_i^2 = (v_i, v_i)$. From this we learn:

Exercise 1.16.   1. $N : \Gamma_{p,q} \to \mathbb{R}^{\times}$ is a homomorphism.
    2. For $g \in \Gamma_{p,q}$, $N(g) = N(\bar{g})$.

Now let $C\ell^0_{p,q}$, $C\ell^1_{p,q}$ denote the $+1$ and -1 eigenspaces respectively of the order involution so that $C\ell_{p,q} = C\ell^0_{p,q} \oplus C\ell^1_{p,q}$ is a $\mathbb{Z}_2$-graded algebra. We define subgroups $\mathrm{Pin}(p,q)$ and $\mathrm{Spin}(p,q)$ of $\Gamma_{p,q}$ by

$$\mathrm{Pin}(p,q) = \{g \in \Gamma_{p,q} : N(g) = \pm 1\}$$
$$\mathrm{Spin}(p,q) = \mathrm{Pin}(p,q) \cap C\ell^0_{p,q}.$$

Then we have exact sequences:

$$0 \to \mathbb{Z}_2 \to \mathrm{Pin}(p,q) \to \varnothing[p,q] \to 0$$
$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(p,q) \to \mathrm{SO}(p,q) \to 0$$

where $\mathrm{SO}(p,q) = \varnothing[p,q] \cap \mathrm{SL}(p+q, \mathbb{R})$.

The Lie algebra $\varnothing[p,q]$ of $\mathrm{Pin}(p,q)$ is the commutator $[\mathbb{R}^{p,q}, \mathbb{R}^{p,q}] \subset C\ell^0_{p,q}$ which acts on $\mathbb{R}^{p,q}$ by the derivative of $\widetilde{\mathrm{Ad}}$:

$$\xi \cdot v = \xi v - v\tilde{\xi} = [\xi, v]$$

since $\tilde{\xi} = \xi$.

Before leaving these generalities, we record some simple facts that will be useful later on:

Exercise 1.17. For $g \in \Gamma_{p,q}$, $g^t, \bar{g}, g^{-1}$ are all collinear. Deduce:

   1. For $v \in \mathbb{R}^{p,q}_{\times}$, $w \mapsto vwv$ is a symmetric endomorphism of $\mathbb{R}^{p,q}$;
   2. For $d \in \Gamma_{p,q}$, $d^t d \in \mathbb{R}^{\times}$ and $w \mapsto d^t wd$ is a conformal automorphism of $\mathbb{R}^{p,q}$: for $(v, w) \in \mathbb{R}^{p,q}$,

$$(d^t wd, d^t wd) = (d^t d)^2 (v, w).$$

1.3.2. *Vahlen matrices.* We now specialise to the case $(p, q) = (n + 1, 1)$ and arrive at the whole point of our application of Clifford algebras: write $C\ell_n$ for $C\ell_{n,0}$, the Clifford algebra of Euclidean $\mathbb{R}^n$ and contemplate the algebra $C\ell_n(2)$ of 2×2 matrices with entries in $C\ell_n$. I claim that

$$Cl_n(2) \cong C\ell_{n+1,1}.$$

Since both algebras have dimension 2^{n+2}, this amounts to finding a $(n + 2)$-dimensional subspace V of $C\ell_n(2)$ such that:

1. $v^2 = -Q(v)I$ for all $v \in V$ where I is the unit (identity matrix) in $C\ell_n(2)$ and Q is a quadratic form of signature $(n + 1, 1)$;
2. V generates $C\ell_n(2)$.

For this, we take

$$V = \left\{ \begin{pmatrix} x & \lambda \\ \mu & -x \end{pmatrix} : x \in \mathbb{R}^n, \lambda, \mu \in \mathbb{R} \right\}$$

and observe that

$$\begin{pmatrix} x & \lambda \\ \mu & -x \end{pmatrix}^2 = \begin{pmatrix} -x^2 + \lambda\mu & 0 \\ 0 & -x^2 + \lambda\mu \end{pmatrix} = (-x^2 + \lambda\mu)I.$$

Thus we have light-like vectors $v_0, v_\infty \in V$ with $(v_0, v_\infty) = -\frac{1}{2}$ given by

$$v_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \qquad v_\infty = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

and V therefore has an inner product of signature $(n + 1, 1)$.

Exercise 1.18. V generates $C\ell_n(2)$.

This establishes the claim and henceforth we shall write $\mathbb{R}^{n+1,1}$ for $V \subset C\ell_n(2)$.

In fact, we have a little more: the decomposition of $C\ell_n(2)$ into diagonal and off-diagonal matrices gives us a decomposition

$$\mathbb{R}^{n+1,1} = \mathbb{R}^n \oplus \mathbb{R}^{1,1}$$

and fixed light-vectors $v_0, v_\infty \in \mathbb{R}^{1,1}$ lying in a component $\mathbb{L}^+$ of $\mathbb{L}$. Conversely, each such decomposition of $\mathbb{R}^{n+1,1}$ with chosen light-vectors in $\mathbb{R}^{1,1}$ gives us an isomorphism $C\ell_n(2) \cong C\ell_{n+1,1}$.

The distinguished light-vectors v_0, v_∞ give us a ready-made stereographic projection $\mathbb{P}(\mathbb{L}) \setminus \langle v_\infty \rangle \to \mathbb{R}^n = \langle v_0, v_\infty \rangle^\perp$. Indeed,

$$E_{v_\infty} = \{v \in \mathbb{L} : (v, v_\infty) = -\tfrac{1}{2}\} = \left\{ \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} : x \in \mathbb{R}^n \right\}$$

and the stereo-projection of (1.1) reads

$$\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} \mapsto \begin{pmatrix} x & 0 \\ 0 & -x \end{pmatrix} = x \in \mathbb{R}^n$$

with inverse

$$x \mapsto \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}.$$

The various (anti-)involutions on $C\ell_n(2)$ are readily identified:

Exercise 1.19. For $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C\ell_n(2) \cong C\ell_{n+1,1}$,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^- = \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^t = \begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\sim} = \begin{pmatrix} \tilde{a} & -\tilde{b} \\ -\tilde{c} & \tilde{d} \end{pmatrix}$$

Now let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{n+1,1}$. Since $N(g) = g\bar{g} \in \mathbb{R}^\times$ we deduce from

$$N(g) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d^t & -b^t \\ -c^t & a^t \end{pmatrix} = \begin{pmatrix} ad^t - bc^t & ba^t - ab^t \\ cd^t - dc^t & da^t - cb^t \end{pmatrix} \in \mathbb{R}^\times I$$

that

$$ad^t - bc^t = da^t - cb^t \in \mathbb{R}^\times$$

$$cd^t = dc^t \qquad ab^t = ba^t.$$

Moreover, $N(g) = N(\bar{g})$ gives

$$ad^t - bc^t = d^t a - b^t c$$

$$c^t a = a^t c \qquad d^t b = b^t d.$$

These are all necessary conditions for the matrix g to lie in $\Gamma_{n+1,1}$. The full story is the content of Vahlen's theorem:

Theorem 1.4. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_{n+1,1}$ *if and only if* $a, b, c, d \in \Gamma_n \cup \{0\}$ *with*

1. $ad^t - bc^t \in \mathbb{R}^\times$;
2. $ac^t, bd^t, a^t b, c^t d \in \mathbb{R}^n$.

Exercise 1.20. For $a, c \in \Gamma_n \cup \{0\}$, $ac^t \in \mathbb{R}^n$ if and only if $a^t c \in \mathbb{R}^n$. Then take transposes to get $ca^t, c^t a \in \mathbb{R}^n$ also.

We now restrict attention to the open subgroup $\mathrm{SL}(\Gamma_n)$ of $\mathrm{Pin}(n+1,1)$ given by

$$\mathrm{SL}(\Gamma_n) = N^{-1}\{1\}.$$

This has two components $\mathrm{SL}(\Gamma_n) \cap C\ell^0_{n+1,1}$ and $\mathrm{SL}(\Gamma_n) \cap C\ell^1_{n+1,1}$ and double covers $\mathrm{O}^+(n+1,1) \cong \mathrm{Möb}(n)$.

Remark. $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in C\ell^0_{n+1,1}$ if and only if $a, d \in C\ell^0_n$ and $b, c \in C\ell^1_n$.

Our formalism gives a beautiful description of the action of $\mathrm{SL}(\Gamma_n)$ on $\mathbb{R}^n$ by linear fractional transformations: $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma_n)$ induces a conformal diffeomorphism of $\mathbb{R}^n \cup \{\infty\}$ which we denote by $x \mapsto g \cdot x$. To compute this, note that $g\bar{g} = 1$ whence $\tilde{g}^{-1} = g^t$ so that

$$\widetilde{\mathrm{Ad}}(g)v = gvg^t.$$

Embedding $\mathbb{R}^n$ as usual into L^+ by inverse stereo-projection,

$$x \mapsto \begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}$$

we have

$$\widetilde{\mathrm{Ad}}(g)\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix}\begin{pmatrix} \bar{d} & \bar{b} \\ \bar{c} & \bar{a} \end{pmatrix}$$

$$= \begin{pmatrix} (ax+b)\overline{(cx+d)} & (ax+b)\overline{(ax+b)} \\ (cx+d)\overline{(cx+d)} & (cx+d)\overline{(ax+b)} \end{pmatrix}.$$

Exercise 1.21. For $x \in \mathbb{R}^n$ and $c, d \in \Gamma_n \cup \{0\}$, $cx+d \in \Gamma_n \cup \{0\}$ and, in particular, $(cx+d)\overline{(cx+d)} \in \mathbb{R}$.

In the case at hand, either $cx + d = 0$ in which case

$$\widetilde{\mathrm{Ad}}(g)\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} = (ax+b)\overline{(ax+b)}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in \langle v_\infty \rangle$$

so that $g \cdot x = \infty$ or else

$$\widetilde{\mathrm{Ad}}(g)\begin{pmatrix} x & -x^2 \\ 1 & -x \end{pmatrix} = (cx+d)\overline{(cx+d)}\begin{pmatrix} (ax+b)(cx+d)^{-1} & * \\ 1 & * \end{pmatrix}$$

with stereo-projection $(ax+b)(cx+d)^{-1} \in \mathbb{R}^n$.

Exercise 1.22. Show that

$$\widetilde{\mathrm{Ad}}(g)v_\infty = \begin{cases} c\bar{c}\begin{pmatrix} ac^{-1} & * \\ 1 & * \end{pmatrix} & \text{if } c \neq 0; \\ a\bar{a}v_\infty & \text{if } c = 0. \end{cases}$$

Otherwise said, $g \cdot \infty = ac^{-1} \in \mathbb{R}^n \cup \{\infty\}$.

To summarise: the action of g as a conformal diffeomorphism of $\mathbb{R}^n \cup \{\infty\}$ is given by

$$g \cdot x = (ax + b)(cx + d)^{-1}.$$

Example. $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}(\Gamma_n)$ acts by $x \mapsto -x^{-1} = x/\|x\|^2$: this is inversion in the unit sphere.

Having understood the groups involved, let us briefly consider the Lie algebra $\varnothing = [\mathbb{R}^{n+1,1}, \mathbb{R}^{n+1,1}]$.

Exercise 1.23. Show that

$$\varnothing = \left\{ \begin{pmatrix} \xi & x \\ y & -\xi^t \end{pmatrix} : x, y \in \mathbb{R}^n, \xi \in [\mathbb{R}^n, \mathbb{R}^n] \oplus \mathbb{R} \right\}$$

Note that the decomposition of $\varnothing$ into diagonal and off-diagonal pieces,

$$\varnothing = \mathfrak{k} \oplus \mathfrak{p},$$

$\mathfrak{k} = [\mathbb{R}^n, \mathbb{R}^n] \oplus \mathbb{R}$, $\mathfrak{p} = \mathbb{R}^n \oplus \mathbb{R}^n$, is a symmetric decomposition. Indeed, $\mathfrak{k}$, $\mathfrak{p}$ are, respectively the $+1$ and -1 eigenspaces of the involution in $\mathrm{O}^+(n+1,1)$ which is $+1$ on $\mathbb{R}^{1,1}$ and -1 on $\mathbb{R}^n$. The corresponding symmetric space will play a starring role in Section 3.

Remark. We have confined our exposition to the case $\mathbb{R}^{n+1,1}$ of direct relevance to the theory we wish to develop. However, the analogous theory holds for any $\mathbb{R}^{p+1,q+1}$. Again $C\ell_{p+1,q+1} \cong C\ell_{p,q}(2)$ and the analog of Vahlen's theorem identifies $\Gamma_{p+1,q+1}$ (with the refinement that $\Gamma_n \cup \{0\}$ is replaced by the monoid generated by all elements of $\mathbb{R}^{p,q}$ whether invertible or not). Again, the projective light cone in $\mathbb{R}^{p+1,q+1}$ is the conformal compactification of $\mathbb{R}^{p,q}$ and we arrive at a description

of the conformal group of $\mathbb{R}^{p,q}$ in terms of 2×2 matrices with entries in $C\ell_{p,q}$ and linear fractional transformations [58]. It is hard not to hope that the methods elaborated here may have applications in this more general setting. A good test case for this would be to take $(p,q) = (3,1)$ where these ideas describe the symmetry group $\varnothing[4,2]$ of Lie sphere geometry [19].

1.3.3. *Moving frames.* We have now arrived at the model of conformal geometry with which we shall work for the rest of this paper. Let us summarise this picture: we work with the "Euclidean" model $\mathbb{R}^n \cup \{\infty\}$ of the conformal n-sphere using stereo-projection (1.1) to identify $\mathbb{R}^n \cup \{\infty\}$ with $E_{v_\infty} \cup \{v_\infty\} \subset L^+$ and so, via π, with $\mathbb{P}(L)$. The projective action of $\mathrm{SL}(\Gamma_n)$ on $\mathbb{P}(L)$ induces an action[9] on $E_{v_\infty} \cup \{v_\infty\}$ and so on $\mathbb{R}^n \cup \{\infty\}$ by conformal diffeomorphisms. We have seen that this action on $\mathbb{R}^n \cup \{\infty\}$ is given by

(1.5a) $$g \cdot x = (ax + b)(cx + d)^{-1}$$

(1.5b) $$g \cdot \infty = ac^{-1},$$

for $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(\Gamma_n)$.

In what follows, we shall study maps $f : M \to \mathbb{R}^n$ and also pairs of maps $f, \hat{f} : M \to \mathbb{R}^n$. A useful technique for this is the method of the moving frame: a *frame* for f is a map $F : M \to \mathrm{SL}(\Gamma_n)$ such that

$$f = F \cdot 0.$$

Example. $F = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ frames f.

Similarly, a frame for the (ordered) pair $(f, \hat{f})$ is a map $F : M \to \mathrm{SL}(\Gamma_n)$ such that

$$f = F \cdot 0 \qquad \hat{f} = F \cdot \infty.$$

In this case, with $F = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we have

$$f = bd^{-1} \qquad \hat{f} = ac^{-1}$$

so that

(1.6) $$F = \begin{pmatrix} \hat{f}c & fd \\ c & d \end{pmatrix}$$

and the determinant condition of Theorem 1.4 reads

(1.7) $$(\hat{f} - f)cd^t = 1.$$

In fact, once this condition is satisfied, F defined by (1.6) automatically satisfies the remaining conditions of Vahlen's Theorem and so lies in $\mathrm{SL}(\Gamma_n)$:

Exercise 1.24. If f and $\hat{f}$ never coincide[10] and $c, d \in \Gamma_n$ satisfy (1.7), then F defined by (1.6) lies in $\mathrm{SL}(\Gamma_n)$.

Example. The pair $(f, \hat{f})$ is framed by $\begin{pmatrix} \hat{f}(\hat{f} - f)^{-1} & f \\ (\hat{f} - f)^{-1} & 1 \end{pmatrix}$.

---

[9]Thus the action on $E_{v_\infty} \cup \{v_\infty\}$ is the linear action $\widetilde{\mathrm{Ad}}$ on L followed by rescaling to ensure that the end result lies in $E_{v_\infty} \cup \{v_\infty\}$.

[10]This is a necessary condition for the pair to be framed since $g \cdot 0 \neq g \cdot \infty$ for any $g \in \mathrm{SL}(\Gamma_n)$.

The point of using frames is that maps into a group are essentially determined by their derivative. For $F : M \to \mathrm{SL}(\Gamma_n)$, consider the *Maurer–Cartan* form of F given by $B = F^{-1}\mathrm{d}F$: this is a 1-form with values in ø. Differentiating the $C\ell_n(2)$-valued equation

$$\mathrm{d}F = FB$$

gives the Maurer–Cartan equations

(1.8) $$\mathrm{d}B + B \wedge B = 0$$

where multiplication in $C\ell_n(2)$ is used to multiply the coefficients of B in $B \wedge B$.

Conversely, given such a 1-form B satisfying (1.8), we can locally[11] integrate [63] to find a map $F : M \to \mathrm{SL}(\Gamma_n)$ with $F^{-1}\mathrm{d}F = B$ which is unique up to left multiplication by constants in $\mathrm{SL}(\Gamma_n)$. The Maurer–Cartan equations (1.8) amount to "structure equations" for the immersions framed by F.

Exercise 1.25. Set $g = \hat{f} - f$ and put $F = \begin{pmatrix} \hat{f}g^{-1} & f \\ g^{-1} & 1 \end{pmatrix}$: we have seen that F frames $(f, \hat{f})$. Show that

$$F^{-1}\mathrm{d}F = \begin{pmatrix} (\mathrm{d}f)g^{-1} & \mathrm{d}f \\ -g^{-1}(\mathrm{d}\hat{f})g^{-1} & -g^{-1}\mathrm{d}f \end{pmatrix}.$$

1.4. Exterior calculus on $\Omega \otimes C\ell_n$ and applications.

1.4.1. *Clifford algebra valued differential forms.* Let M be a manifold and Ω the exterior algebra of differential forms on M. Consider the space $\Omega \otimes C\ell_n$ of $C\ell_n$-valued forms on M. Since $C\ell_n$ is an associative algebra, we may extend exterior multiplication to $\Omega \otimes C\ell_n$ by using the product in $C\ell_n$ to multiply coefficients. Thus for monomials $a\omega_1, b\omega_2 \in \Omega \otimes C\ell_n$ with $a, b : M \to C\ell_n$, $\omega_i \in \Omega$:

$$a\omega_1 \wedge b\omega_2 = (ab)\omega_1 \wedge \omega_2.$$

In particular, for $f, g \in \Omega^0 \otimes C\ell_n$, that is, $f, g : M \to C\ell_n$, the exterior product is just pointwise multiplication.

Similarly, we extend the exterior derivative by

$$\mathrm{d}(a\omega) = \mathrm{d}a \wedge \omega + a\mathrm{d}\omega.$$

Since $C\ell_n$ is not, in general, commutative, exterior multiplication on $\Omega \otimes C\ell_n$ is no longer super-commutative:

$$\alpha \wedge \beta \neq \pm\beta \wedge \alpha.$$

However, it is not difficult to establish:

Proposition 1.5. *For $\alpha \in \Omega^p \otimes C\ell_n$, $\beta \in \Omega^q \otimes C\ell_n$ and $f \in \Omega^0 \otimes C\ell_n$:*

  1. $\alpha f \wedge \beta = \alpha \wedge f\beta$ *(this is a special case of the associativity of $\wedge$).*
  2. $\tilde{\alpha} \wedge \tilde{\beta} = (\alpha \wedge \beta)^{\sim}$.
  3. $\alpha^t \wedge \beta^t = (-1)^{pq}(\beta \wedge \alpha)^t$.
  4. $\bar{\alpha} \wedge \bar{\beta} = (-1)^{pq}\overline{(\beta \wedge \alpha)}$.
  5. $\mathrm{d}(\alpha \wedge \beta) = \mathrm{d}\alpha \wedge \beta + (-1)^p\alpha \wedge \mathrm{d}\beta$.
  6. $\mathrm{d}^2 = 0$.

Exercise 1.26. If $g : M \to C\ell_n^{\times}$, differentiate $gg^{-1} = 1$ to conclude:

$$\mathrm{d}g^{-1} = -g^{-1}(\mathrm{d}g)g^{-1}.$$

---

[11]That is, on simply connected subdomains of M.

This exterior calculus will be our main computational tool for much of these lectures.

1.4.2. *A lemma on commuting forms in $\Omega^1 \otimes \mathbb{R}^n$.* With an eye to a basic application to isothermic surfaces, we prove:

Lemma 1.6. *Let V be a real vector space with $\dim V \geq 2$ and $\alpha, \beta : V \to \mathbb{R}^n$ non-zero linear maps with α injective. Consider $\alpha \wedge \beta : \bigwedge^2 V \to C\ell_n$. Then*

$$(1.9) \qquad\qquad \alpha \wedge \beta = 0$$

if and only if the following conditions are satisfied:

1. $\dim V = 2$;
2. *There is $\lambda \in \mathbb{R}^+$ such that $(\beta, \beta) = \lambda(\alpha, \alpha)$;*
3. $\operatorname{Im} \alpha = \operatorname{Im} \beta$;
4. $\det(\alpha^{-1} \circ \beta) < 0$.

Thus α and β have the same image, induce conformally equivalent inner products on V but opposite orientations.

Proof. Suppose first that $\alpha \wedge \beta = 0$. Choose an orthonormal basis $e_1, \ldots, e_n$ of $\mathbb{R}^n$ so that $\operatorname{Im} \alpha = \langle e_1, \ldots, e_m \rangle$ and a basis $\omega_1, \ldots, \omega_m$ of V^* so that

$$\alpha = \sum_{i \leq m} e_i \otimes \omega_i.$$

Write

$$\beta = \sum_{j \leq m} e_j \otimes \eta_j + \sum_{l > m} e_l \otimes \eta_l$$

for some $\eta_1, \ldots, \eta_n \in V^*$. Then (1.9) reads

$$(1.10) \qquad \sum_{i,j \leq m} e_i e_j \omega_i \wedge \eta_j + \sum_{\substack{i \leq m \\ l > m}} e_i e_l \omega_i \wedge \eta_l = 0.$$

The elements $1, e_i e_j \ (i < j)$ are linearly dependent in $C\ell_n$ while $e_i e_j = -e_j e_i$, for $i \neq j$, whence taking coefficients in (1.10) gives:

$$(1.11a) \qquad \sum_{i \leq m} \omega_i \wedge \eta_i = 0$$

$$(1.11b) \qquad \omega_i \wedge \eta_j = \omega_j \wedge \eta_i \qquad\qquad \text{for } 1 \leq i < j \leq m.$$

$$(1.11c) \qquad \omega_i \wedge \eta_l = 0 \qquad\qquad \text{for } 1 \leq i \leq m, l > m.$$

From (1.11c), we see that each $\eta_l = 0$, for $l > m$, so that $\operatorname{Im} \beta \subset \operatorname{Im} \alpha$.

Applying the Cartan Lemma to (1.11a), we get, for each $j \leq m$,

$$\eta_j = \sum_{i \leq m} a_{ji} \omega_i$$

with $a_{ij} = a_{ji}$. Thus, (1.11b) becomes, for fixed $i, j \leq m$,

$$\sum_k a_{jk} \omega_i \wedge \omega_k = \sum_k a_{ik} \omega_j \wedge \omega_k$$

from which we conclude that $a_{ik} = 0$ whenever $k \neq i, j$ and $a_{ii} = -a_{jj}$. If we can choose i, j, k all distinct we quickly conclude that all $a_{ij} = 0$ so that $\beta = 0$: a contradiction. We must therefore have $\dim V = 2$ and

$$\eta_1 = a_{11}\omega_1 + a_{12}\omega_2$$
$$\eta_2 = a_{12}\omega_1 - a_{11}\omega_2$$

with a_{11} and a_{12} not both zero. We now have

$$(\beta, \beta) = \eta_1^2 + \eta_2^2$$
$$= (a_{11}^2 + a_{12}^2)(\omega_1^2 + \omega_2^2) = (a_{11}^2 + a_{12}^2)(\alpha, \alpha)$$

and

$$\det(\alpha^{-1} \circ \beta) = \det(a_{ij}) = -a_{11}^2 - a_{12}^2 < 0.$$

The converse is more direct: let $\dim V = 2$ and choose v_1, v_2 an orthonormal basis of V with respect to (α, α) and set $Z = v_1 + iv_2 \in V^{\mathbb{C}}$. Thus $(\alpha(Z), \alpha(Z)) = 0$. Now $\operatorname{Im} \alpha = \operatorname{Im} \beta$ gives $\beta(Z) \in \langle \alpha(Z), \alpha(\bar{Z}) \rangle$ while $(\beta, \beta) = \lambda(\alpha, \alpha)$ forces $(\beta(Z), \beta(Z)) = 0$ so that $\beta(Z)$ is parallel to either $\alpha(Z)$ or $\alpha(\bar{Z})$. Finally, $\det(\alpha^{-1} \circ \beta) < 0$ forces the second possibility to hold so that there is $\mu \in \mathbb{C}$ such that

$$\beta(Z) = \mu\alpha(\bar{Z}), \quad \beta(\bar{Z}) = \bar{\mu}\alpha(Z).$$

Then

$$\alpha \wedge \beta(Z, \bar{Z}) = \alpha(Z)\beta(\bar{Z}) - \alpha(\bar{Z})\beta(Z)$$
$$= \bar{\mu}\alpha(Z)^2 - \mu\alpha(\bar{Z})^2 = 0$$

since $\alpha(Z)^2 = -(\alpha(Z), \alpha(Z)) = 0$ and similarly for $\alpha(\bar{Z})^2$. $\qquad\square$

1.4.3. *More on sphere congruences.* Let $f, \hat{f} : M \to \mathbb{R}^n$ be immersions of a k-dimensional manifold. We give a simple analytic condition for f and $\hat{f}$ to envelope the same sphere congruence:

Proposition 1.7. *Let $g = \hat{f} - f$. Then f and $\hat{f}$ envelope the same sphere congruence if and only if*

$$(1.12) \qquad\qquad \operatorname{Im} d\hat{f} = \operatorname{Im} g df g^{-1}.$$

Proof. The hypothesis (1.12) means that $\operatorname{Im} d\hat{f} = \operatorname{Im} \rho_g df$ where $\rho_g = \widetilde{\operatorname{Ad}}(g)$ is reflection in the hyperplane orthogonal to g.

Now fix $p \in M$ and restrict attention to a $(k+1)$-dimensional affine space[12] containing $\hat{f}(p)$ and $df(T_pM) + f(p)$. Certainly, any k-sphere (or k-plane) tangent to f and $\hat{f}$ at p lie in this space. Now any k-sphere containing $f(p)$ and $\hat{f}(p)$ must have centre on the hyperplane orthogonal to $g(p)$ through $\frac{1}{2}(f(p) + \hat{f}(p))$ and so is stable under reflection in this hyperplane (which interchanges $f(p)$ and $\hat{f}(p)$). Moreover, there is a unique k-sphere (or possibly k-plane) of this kind whose tangent space at $f(p)$ is $df(T_pM)$. The tangent space to this sphere at $\hat{f}(p)$ is therefore $\rho_g df(T_pM)$ which is tangent to $\hat{f}$ at p if and only if $\rho_g df(T_pM) = d\hat{f}(T_pM)$. $\qquad\square$

Exercise 1.27. In the situation of Proposition 1.7, if N is the unit normal of f pointing towards the centres of the sphere congruence, then the radii r of the spheres are given by

$$1/r = -2(g^{-1}, N).$$

Remark. ρ_g restricts in this case to an isomorphism between the normal bundles of f and $\hat{f}$ which, since ρ_g is an isometry, is parallel for the normal connections on those bundles.

For future use, we record:

---

[12]This space is uniquely determined unless $g(p) \in df(T_pM)$.

Exercise 1.28. With $\operatorname{Im} d\hat{f} = \operatorname{Im} g df g^{-1}$ and N normal to f,

$$d(gNg^{-1}) = g\big(dN - 2(g^{-1}, N)g^{-1}d\hat{f}g - 2(g^{-1}, N)df\big)g^{-1}.$$

We conclude our present discussion of sphere congruences by considering a degenerate case that we wish to exclude from further discussions:

Definition. A sphere congruence $S : M \to G^+_{n-k}(\mathbb{R}^{n+1,1})$ is *full* if there is no fixed hyperplane $\Pi \subset \mathbb{R}^{n+1,1}$ containing all the $(n-k)$-planes $S(p), p \in M$.

Let us contemplate non-full congruences. The geometry of this condition depends on the signature of the inner product when restricted to Π:

1. If Π has signature $(n, 1)$, all spheres in the congruence cut the hypersphere determined by the space-like line $\Pi^\perp$ orthogonally.
2. If $\Pi^\perp \in \mathbb{P}(\mathrm{L})$ (that is, Π has signature $(n, 0)$) then all spheres in the congruence contain the point $\Pi^\perp$.
3. If Π is space-like then all spheres in the congruence lie totally geodesically in the round n-sphere determined by a unit time-like vector in $\Pi^\perp$.

Now restrict attention to the case where Π has non-degenerate inner product. Then our non-full sphere congruence is stable under the Möbius transformation R induced by reflection in Π. As a consequence, if f envelopes the congruence, so does $R \circ f$. We give an analytic condition, refining that of Proposition 1.7, for two enveloping surfaces to arise this way.

Proposition 1.8. $(f, \hat{f})$ *envelope a non-full sphere congruence with* $\hat{f} = R \circ f$ *if and only if there is a function* $\mu : M \to \mathbb{R}^\times$ *such that*

$$(1.13) \qquad\qquad g^{-1}d\hat{f}g^{-1} = \mu df.$$

Proof. Suppose first that $\hat{f} = R \circ f$ where R is the Möbius transformation induced by reflection in a non-degenerate hyperplane $\Pi \subset \mathbb{R}^{n+1,1}$. Let $F : M \to \mathrm{SL}(\Gamma_n)$ be the frame of $(f, \hat{f})$ given by

$$F = \begin{pmatrix} \hat{f}g^{-1} & f \\ g^{-1} & 1 \end{pmatrix}$$

(cf. Exercise 1.25) and fix $v \in \Pi^\perp$ with $v^2 = \pm 1$. Up to a scaling, $\widetilde{\mathrm{Ad}}(F)v_\infty$ is the reflection in Π of $\widetilde{\mathrm{Ad}}(F)v_0$ whence $v \in \langle \widetilde{\mathrm{Ad}}(F)v_0, \widetilde{\mathrm{Ad}}(F)v_\infty \rangle$ so that

$$(1.14) \qquad\qquad v = \mathrm{Ad}(F) \begin{pmatrix} 0 & \pm e^{-u} \\ e^u & 0 \end{pmatrix},$$

for some $u : M \to \mathbb{R}$. Since v is constant, we have

$$0 = \widetilde{\mathrm{Ad}}(F^{-1})dv$$

$$= d\begin{pmatrix} 0 & \pm e^{-u} \\ e^u & 0 \end{pmatrix} + \left[F^{-1}dF, \begin{pmatrix} 0 & \pm e^{-u} \\ e^u & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} 0 & \mp e^{-u}du \\ e^u du & 0 \end{pmatrix} + \left[\begin{pmatrix} (df)g^{-1} & df \\ -g^{-1}(d\hat{f})g^{-1} & -g^{-1}df \end{pmatrix}, \begin{pmatrix} 0 & \mp e^{-u}du \\ e^u du & 0 \end{pmatrix} \right]$$

$$= \begin{pmatrix} e^u df \pm e^{-u}g^{-1}(d\hat{f})g^{-1} & \mp e^{-u}(du - dfg^{-1} - g^{-1}df) \\ e^u(du - dfg^{-1} - g^{-1}df) & \mp e^{-u}g^{-1}(d\hat{f})g^{-1} - e^u df \end{pmatrix}$$

where we have used Exercise 1.25 to compute $F^{-1}dF$.

Thus we have two equations

(1.15a) $$e^{2u}\mathrm{d}f = \mp g^{-1}\mathrm{d}\hat{f}g^{-1}$$

(1.15b) $$\mathrm{d}u = \{g^{-1}, \mathrm{d}f\},$$

where $\{\,,\,\}$ is the anti-commutator in $C\ell_n$. The first of these is our desired equation (1.13) with

(1.16) $$\mu = \pm e^{2u}.$$

Conversely, if (1.13) holds, define u by (1.16) so that (1.15a) holds and define v by (1.14). At each point $p \in M$, $v(p) \in \langle \widetilde{\mathrm{Ad}}(F(p))v_0, \widetilde{\mathrm{Ad}}(F(p))v_\infty \rangle$ so that the enveloping sphere at p is defined by a $(n-k)$-plane lying in the hyperplane $\langle v(p) \rangle^\perp$. Moreover, reflection in this hyperplane permutes the light-lines spanned by $\widetilde{\mathrm{Ad}}(F(p))v_0$ and $\widetilde{\mathrm{Ad}}(F(p))v_\infty$ so that $\hat{f}(p) = R_p(f(p))$ where R_p is the corresponding Möbius transformation. We will therefore be done if we can show that v is constant which, since (1.15a) holds by construction, amounts to establishing (1.15b). However, differentiating (1.15a) gives

$$
\begin{aligned}
2e^{2u}\mathrm{d}u \wedge \mathrm{d}f &= \pm g^{-1}\mathrm{d}gg^{-1} \wedge \mathrm{d}\hat{f}g^{-1} \mp g^{-1}\mathrm{d}\hat{f} \wedge g^{-1}\mathrm{d}gg^{-1} \\
&= \mp g^{-1}\mathrm{d}f \wedge g^{-1}\mathrm{d}\hat{f}g^{-1} \pm g^{-1}\mathrm{d}\hat{f}g^{-1} \wedge \mathrm{d}fg^{-1} \\
&= e^{2u}g^{-1}\mathrm{d}f \wedge \mathrm{d}f - e^{2u}\mathrm{d}f \wedge \mathrm{d}fg^{-1} \\
&= e^{2u}(g^{-1}\mathrm{d}f \wedge \mathrm{d}f + \mathrm{d}fg^{-1} \wedge \mathrm{d}f - \mathrm{d}f \wedge g^{-1}\mathrm{d}f - \mathrm{d}f \wedge \mathrm{d}fg^{-1}) \\
&= 2e^{2u}\{g^{-1}, \mathrm{d}f\} \wedge \mathrm{d}f.
\end{aligned}
$$

Equation (1.15b) follows immediately and the proof is complete. $\square$

2. ISOTHERMIC SURFACES: CLASSICAL THEORY

2.1. **Isothermic surfaces and their duals.** Let $f : M \to \mathbb{R}^n$ be an immersion of a surface M. We begin with a problem studied by Christoffel [21] for $n = 3$ and Palmer [56] for n arbitrary: under what conditions is there a second immersion $f^c : M \to \mathbb{R}^n$, a *dual surface* of f, such that:

1. f and f^c have parallel tangent planes: $\mathrm{d}f(T_xM) = \mathrm{d}f^c(T_xM)$, for all $x \in M$;
2. f and f^c induce conformally equivalent metrics on M:

$$(\mathrm{d}f, \mathrm{d}f) = \lambda(\mathrm{d}f^c, \mathrm{d}f^c),$$

   for some $\lambda : M \to \mathbb{R}^+$.
3. $\mathrm{d}f^{-1} \circ \mathrm{d}f^c : TM \to TM$ is orientation-reversing: $\det(\mathrm{d}f^{-1} \circ \mathrm{d}f^c) < 0$.

In view of Lemma 1.6, these conditions have a compact formulation in our Clifford algebra formalism: viewing $\mathrm{d}f$ and $\mathrm{d}f^c$ as $C\ell_n$-valued 1-forms, they amount to

$$\mathrm{d}f \wedge \mathrm{d}f^c = 0.$$

This motivates our main definition:

Definition. An immersion $f : M \to \mathbb{R}^n$ is *isothermic* if there is a non-constant map $f^c : M \to \mathbb{R}^n$ such that

(2.1) $$\mathrm{d}f \wedge \mathrm{d}f^c = 0.$$

Note that, away from the zeros of $\mathrm{d}f^c$ (about which more below), f^c is a dual surface of f and is itself isothermic with dual surface f since

$$0 = (\mathrm{d}f \wedge \mathrm{d}f^c)^t = -\mathrm{d}f^c \wedge \mathrm{d}f.$$

Example. For $n = 4$, $C\ell_4 = \mathbb{H}(2)$ with $\mathbb{R}^4 = \mathbb{H}$ embedded in $\mathbb{H}(2)$ via

$$q \mapsto \begin{pmatrix} 0 & q \\ -\bar{q} & 0 \end{pmatrix}.$$

Then, viewing $\mathrm{d}f$ and $\mathrm{d}f^c$ as $\mathbb{H}$-valued 1-forms, equation (2.1) reads

$$\mathrm{d}f \wedge \mathrm{d}\bar{f}^c = 0 = \mathrm{d}\bar{f} \wedge \mathrm{d}f^c$$

which is the characterisation of isothermic surfaces in $\mathbb{R}^4$ given by Hertrich-Jeromin–Pedit [44].

Let $f : M \to \mathbb{R}^n$ be isothermic with dual f^c and equip M with the conformal structure induced by f. Define a quadratic differential $Q_f : \otimes^2 T^{1,0}M \to \mathbb{C}$ by $Q_f = (\mathrm{d}f, \mathrm{d}f^c)^{2,0}$.

Lemma 2.1. *Q_f is a holomorphic quadratic differential.*

Proof. Choose a holomorphic coordinate z on M. We must show that

$$(f_z, f_z^c)_{\bar{z}} = 0.$$

As in Lemma 1.6, there is a function μ with

$$f_z^c = \mu f_{\bar{z}}, \qquad f_{\bar{z}}^c = \bar{\mu} f_z$$

so that

$$\begin{aligned}
(f_z, f_z^c)_{\bar{z}} &= (f_{z\bar{z}}, f_z^c) + (f_z, f_{z\bar{z}}^c) \\
&= \mu(f_{z\bar{z}}, f_{\bar{z}}) + \left(f_z, (\bar{\mu} f_z)_z\right) \\
&= \tfrac{1}{2}\mu(f_{\bar{z}}, f_{\bar{z}})_z + \bar{\mu}_z(f_z, f_z) + \tfrac{1}{2}\bar{\mu}(f_z, f_z)_z = 0
\end{aligned}$$

since both $(f_{\bar{z}}, f_{\bar{z}})$ and (f_z, f_z) vanish by the conformality of f. $\qquad\square$

Corollary 2.2. *Q_f and so $\mathrm{d}f^c$ vanish on at most a discrete set.*

Thus f^c is at worst a branched conformal immersion.

We have now seen that an isothermic immersion $f : M \to \mathbb{R}^n$ equips M with a conformal structure and a non-zero holomorphic quadratic differential $Q = Q_f \in \Gamma(\otimes^2 T^{1,0}M)$. Otherwise said, (M, Q) is a *polarised Riemann surface* in the sense of [43].

Moreover, we can recover $\mathrm{d}f^c$ from f and this data: for any holomorphic coordinate z on M, write $Q = q\mathrm{d}z^2$, $f_z^c = \mu f_{\bar{z}}$ so that

$$q = (f_z, f_z^c) = \mu(f_z, f_{\bar{z}})$$

whence

(2.2) $$f_z^c = q f_{\bar{z}}/(f_z, f_{\bar{z}}).$$

Equation (2.2) can be given an invariant formulation as follows: for any map $g : M \to \mathbb{R}^n$ of a Riemann surface, write

$$\mathrm{d}g = \partial g + \overline{\partial} g$$

where $\partial g \in C^\infty(T^{1,0}M \otimes \mathbb{C}^n)$ and $\overline{\partial} g = \overline{\partial g}$ (thus, locally, $\partial g = g_z \mathrm{d}z$). Then (2.2) reads:

(2.3) $$\partial f^c = \frac{Q\overline{\partial} f}{(\mathrm{d}f, \mathrm{d}f)},$$

where we have used tensor product to multiply powers of $T^{1,0}M$ and $T^{0,1}M$ and contraction to divide them.

To summarise: a conformal immersion f of a polarised Riemann surface (M, Q) is isothermic with $Q_f = Q$ if and only if the 1-form

$$\eta = \frac{1}{(\mathrm{d}f, \mathrm{d}f)}(Q\bar{\partial}f + \overline{Q}\partial f)$$

is exact[13]. Then $\mathrm{d}f^c = \eta$.

To make contact with the classical notion of an isothermic surface, we compute the condition for the 1-form η to be closed: this is

$$\left(\frac{qf_{\bar{z}}}{(f_z, f_{\bar{z}})}\right)_{\bar{z}} = \left(\frac{qf_z}{(f_z, f_{\bar{z}})}\right)_z.$$

A short calculation using the holomorphicity of q and the conformality of f reduces this to

$$(2.4) \qquad\qquad q(f_{\bar{z}\bar{z}})^{\perp} = \bar{q}(f_{zz})^{\perp}$$

where $^{\perp}$ denotes the component in the normal bundle of f. Away from the (isolated) zeros of Q, we may locally choose $z = x + iy$ so that $q = 1$ and then (2.4) amounts to

$$(f_{xy})^{\perp} = 0$$

so that $\partial/\partial x$ and $\partial/\partial y$ diagonalise the shape operator A^N of any normal N to f. We therefore conclude that:

1. All shape operators of f commute so that f has flat normal bundle;
2. x, y are conformal curvature line (CCL) coordinates on M (that is, conformal coordinates with respect to which each second fundamental form is diagonal).

These last two conditions constitute the classical definition of an isothermic surface [18, 27] and in particular, we have proved the following result of Christoffel ($n = 3$) and Palmer:

Theorem 2.3 ([21, 56]). *Let f have flat normal bundle and CCL coordinate $z = x + iy$ with $(\mathrm{d}f, \mathrm{d}f) = e^{2u}\mathrm{d}z\mathrm{d}\bar{z}$. Then the $\mathbb{R}^n$-valued 1-form defined by*

$$\eta = e^{-2u}\left(f_{\bar{z}}\mathrm{d}z + f_z\mathrm{d}\bar{z}\right)$$

is closed and so locally is $\mathrm{d}f^c$ whence f is isothermic with dual f^c.

How unique is the dual of an isothermic surface? Certainly, if f^c is dual to f then so is any $rf^c + k$ for constants $r \in \mathbb{R}^{\times}$ and $k \in \mathbb{R}^n$ and then Q_f becomes rQ_f. With one interesting exception, these are the only possibilities: if f^c and $\tilde{f}^c$ are both duals of f then, for any holomorphic coordinate z, we have a function μ for which

$$f_z^c = \mu\tilde{f}_z^c, \qquad f_{\bar{z}}^c = \bar{\mu}\tilde{f}_{\bar{z}}^c.$$

Taking normal and tangential components of mixed derivatives of $\tilde{f}^c$ gives

$$\mu f_{z\bar{z}}^c = \bar{\mu}f_{z\bar{z}}^c$$
$$\mu_{\bar{z}}f_z^c = \bar{\mu}_z f_{\bar{z}}^c$$

so that μ is holomorphic. Moreover μ is real (and so constant) unless $f_{z\bar{z}}^c = 0$, that is, unless f^c is minimal. In this latter case, for any normal vector field N to f (and so f^c also), we have $(f_{\bar{z}}^c, N_z) = 0$ whence (f_z, N_z) vanishes also and f is totally umbilic. Thus f takes values in a plane or 2-sphere and f^c is a minimal surface in $\mathbb{R}^3$.

We have therefore proved:

---

[13]This latter condition is what Kamberov [47] calls *globally isothermic* when $n = 3$.

Proposition 2.4. *Let $f : M \to \mathbb{R}^n$ be a full[14] isothermic surface. Then the dual f^c of f is unique up to scale and translations unless $n = 3$ and f has image in a 2-sphere.*

An example. Let $f : M \to \mathbb{R}^n$ be an isometric immersion with mean curvature vector $\mathbf{H}$, that is,

$$\mathbf{H} = \tfrac{1}{2} \operatorname{trace} \nabla \mathrm{d} f$$

where ∇ is the connection on $T^*M \otimes f^{-1}T\mathbb{R}^n$ induced by the Levi–Civita connections of M and $\mathbb{R}^n$.

Following Chen [20], a unit normal vector field N of f is said to be an *isoperimetric section* if $(\mathbf{H}, N)$ is constant and a *minimal section* if $(\mathbf{H}, N) = 0$. Of course, when $n = 3$, N is isoperimetric, respectively minimal, if and only if f has constant mean curvature, respectively is minimal, and this motivates the following terminology:

Definition. A surface is said to be a *generalised H-surface* if it admits a parallel isoperimetric section.

Generalised H-surfaces provide a class of examples of isothermic surfaces in view of:

Proposition 2.5. *Let $f : M \to \mathbb{R}^n$ be an immersion and $N : M \to \mathbb{R}^n$ a unit normal vector field not equal[15] to any $rf + k$ for constants $r \in \mathbb{R}$, $k \in \mathbb{R}^n$. Let $\phi = M \to \mathbb{R}$. Then*

1. *ϕN is dual to f if and only if ϕ is constant and N is a parallel minimal section.*
2. *$f + \phi N$ is dual to f if and only if ϕ is constant and N is a parallel isoperimetric section with*

$$(\mathbf{H}, N) = 1/\phi.$$

Proof. For z a holomorphic coordinate on M,

$$(\mathbf{H}, N) = -\frac{(N_z, f_{\bar{z}})}{(f_z, f_{\bar{z}})}.$$

Now $f + \phi N$ is dual to f if and only if $(f + \phi N)_z$ is parallel to $f_{\bar{z}}$ or, equivalently,

(2.5a) $$(\phi_z N, N_1) + \phi(N_z, N_1) = 0$$
(2.5b) $$(f_z, f_{\bar{z}}) + \phi(N_z, f_{\bar{z}}) = 0,$$

for any normal N_1 to f. Taking $N_1 = N$ in (2.5a) gives $\phi_z = 0$ and then (2.5a) asserts that N is parallel while (2.5b) asserts that $\phi = 1/(\mathbf{H}, N)$ so that N is isoperimetric.

The case of minimal N is similar. $\square$

Let us collect some special cases of classical interest:

1. Let $f : M \to \mathbb{R}^3$ have constant mean curvature $H \neq 0$ and Gauss map $N : M \to S^2$ so that $\mathbf{H} = HN$. We see that f is isothermic with parallel dual surface

$$f^c = f + \frac{1}{H} N$$

---

[14]that is, the image of f is not contained in any affine hyperplane.
[15]This is to exclude the case where f has image in a hyper-sphere and N is the normal to that sphere.

which also has constant mean curvature H. Moreover, the Hopf differential of f is $-HQ_f$.

2. If $f : M \to \mathbb{R}^3$ is minimal then f is isothermic with its Gauss map as dual surface and Q_f is the negative of its Hopf differential.

3. Let $N : M \to S^{n-1} \subset \mathbb{R}^n$ be isothermic then Christoffel's formula (2.3) provides a dual surface f for which N is a parallel minimal section. For $n = 3$, this is particularly interesting since *any* conformal map $M \to S^2$ is locally isothermic. Indeed, if $g : \Omega \subset \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ is a meromorphic function, inverse stereo-projection gives us a conformal map $N : M \to S^2$. Moreover, since S^2 is totally umbilic, all directions are curvature directions so that any holomorphic coordinate z on Ω is CCL. Now let f be holomorphic on Ω and set $Q = f\mathrm{d}z^2$. The formula (2.3) for the dual of N now reads

$$N_z^c = \frac{f}{g'}\left(\tfrac{1}{2}(1 - g^2), \tfrac{i}{2}(1 + g^2), g\right)$$

which we recognise as the Weierstrass–Enneper formula for the minimal surface N^c with Hopf differential $-f\mathrm{d}z^2$.

This explains the lack of uniqueness discussed in Proposition 2.4: the Weierstrass–Enneper formula requires a choice of Hopf differential to prescribe a minimal surface with Gauss map N.

4. Let $f : M \to S^3 \subset \mathbb{R}^4$ be minimal with polar surface $f^\perp : M \to S^3$ (thus $f^\perp \perp \langle f, \mathrm{d}f(TM)\rangle$). Then $f^\perp$, viewed as a section of the normal bundle of f in $\mathbb{R}^4$, is certainly a parallel minimal section and so is dual to f in $\mathbb{R}^4$. The symmetry of the situation ensures that $f^\perp$ is minimal in S^3 also.

5. Similarly, if $f : M \to S^3$ has constant mean curvature $H \neq 0$ in S^3 then $f^\perp$ is a parallel isoperimetric section with $(\mathbf{H}, f^\perp) = H$ so that $f + \frac{1}{H}f^\perp$ is dual to f. Further, this parallel surface has constant mean curvature in a sphere of radius $\sqrt{1 + 1/H^2}$.

2.2. Transformations of isothermic surfaces.

A triumph in the classical study of isothermic surfaces was the discovery by Darboux, Bianchi and Calapso [2, 3, 16, 27] of large families of transformations of isothermic surfaces and permutability theorems relating these. In this section, we shall show that this classical transformation theory goes through unchanged in arbitrary co-dimension.

In all that follows, it will be convenient to fix a choice of dual surface up to translation which amounts to fixing Q_f. We therefore fix a polarised Riemann surface (M, Q) and refine our basic definition:

Definition. A conformal immersion $f : (M, Q) \to \mathbb{R}^n$ is *isothermic* if there is a map $f^c : M \to \mathbb{R}^n$ with

$$\mathrm{d}f \wedge \mathrm{d}f^c = 0, \qquad (\mathrm{d}f, \mathrm{d}f^c)^{2,0} = Q.$$

We say that f^c, which is unique up to translation, is the *Christoffel transform* of f.

Thus f^c is a particular choice of scaling for the dual surface of f and, away from its branch points, is isothermic on (M, Q) with Christoffel transform f.

2.2.1. *Conformal invariance.* At first sight, our theory requires the notion of parallel tangent planes and so is purely Euclidean. However, at least locally, our constructions are conformally invariant:

Proposition 2.6. *Let* $f : (M, Q) \to \mathbb{R}^n$ *be isothermic and* $T \in \mathrm{M\ddot{o}b}(n)$. *Then, locally,* $T \circ f$ *is isothermic on* (M, Q).

Proof. It suffices to check that the inversion $f' = -f^{-1} : (M, Q) \to \mathbb{R}^n$ is isothermic. Now $\mathrm{d}f' = f^{-1}\mathrm{d}ff^{-1}$ and we put $\eta = f\mathrm{d}f^c f$. Then

$$\mathrm{d}f' \wedge \eta = f^{-1}\mathrm{d}f \wedge \mathrm{d}f^c f = 0$$
$$(\mathrm{d}f', \eta)^{2,0} = (\mathrm{d}f, \mathrm{d}f^c)^{2,0} = Q$$

while

$$\mathrm{d}\eta = \mathrm{d}f \wedge \mathrm{d}f^c f - f\mathrm{d}f^c \wedge \mathrm{d}f = 0.$$

Thus, locally, $\eta = \mathrm{d}(f')^c$ with $(f')^c$ the Christoffel transform of f'. □

2.2.2. *The Darboux transform.* Inspired by Hertrich-Jeromin–Pedit [44], we begin with a (temporarily) unmotivated definition:

Definition. Let $f, \hat{f} : M \to \mathbb{R}^n$ be non-constant with $g = \hat{f} - f$. Say that $\hat{f}$ is a *Darboux transform* of f, or that $(f, \hat{f})$ are a *Darboux pair*, if

(2.6) $$\mathrm{d}f \wedge g^{-1}\mathrm{d}\hat{f}g^{-1} = 0.$$

Note that, in this case, f is a Darboux transform of $\hat{f}$ also.

Observe that if $(f, \hat{f})$ is a Darboux pair then

$$\mathrm{d}(g^{-1}\mathrm{d}\hat{f}g^{-1}) = \mathrm{d}g^{-1} \wedge \mathrm{d}\hat{f}g^{-1} - g^{-1}\mathrm{d}\hat{f} \wedge \mathrm{d}g^{-1}$$
$$= -g^{-1}\mathrm{d}gg^{-1} \wedge \mathrm{d}\hat{f}g^{-1} + g^{-1}\mathrm{d}\hat{f} \wedge g^{-1}\mathrm{d}gg^{-1}$$
$$= g^{-1}\mathrm{d}fg^{-1} \wedge \mathrm{d}\hat{f}g^{-1} - g^{-1}\mathrm{d}\hat{f}g^{-1} \wedge \mathrm{d}fg^{-1} = 0$$

in view of (2.6) and its transpose. Thus, locally, $g^{-1}\mathrm{d}\hat{f}g^{-1} = \mathrm{d}f^c$ for f^c a dual surface to f. Thus f is isothermic and so is $\hat{f}$ with dual given by $\mathrm{d}\hat{f}^c = g^{-1}\mathrm{d}fg^{-1}$.

Moreover,
$$(\mathrm{d}f^c, \mathrm{d}f^c) = (g^{-1}\mathrm{d}\hat{f}g^{-1}, g^{-1}\mathrm{d}\hat{f}g^{-1}) = g^{-4}(\mathrm{d}\hat{f}, \mathrm{d}\hat{f})$$
so that f and f^c induce the same conformal structure on M. Further, using Exercise 1.17,

$$(\mathrm{d}f, \mathrm{d}f^c) = (\mathrm{d}f, g^{-1}\mathrm{d}\hat{f}g^{-1}) = (g^{-1}\mathrm{d}fg^{-1}, \mathrm{d}\hat{f}) = (\mathrm{d}\hat{f}^c, \mathrm{d}\hat{f})$$

so that $Q_f = Q_{\hat{f}}$ and f and $\hat{f}$ induce the same polarisation on M also.

To summarise:

Theorem 2.7. *If $(f, \hat{f})$ are a Darboux pair, then f and $\hat{f}$ are isothermic on the same polarised Riemann surface.*

Now let (M, Q) be a polarised Riemann surface and let $f : (M, Q) \to \mathbb{R}^n$ be isothermic with Christoffel transform f^c. We seek Darboux transforms of f. If $\hat{f} = f + g$ is a Darboux transform then, from (2.6), we have that $g^{-1}\mathrm{d}\hat{f}g^{-1}$ is the derivative of a dual surface to f so that, for some $r \in \mathbb{R}^\times$,

$$\mathrm{d}\hat{f} = \mathrm{d}f + \mathrm{d}g = rg\mathrm{d}f^c g.$$

We rearrange this into a Riccati equation for g:

(2.7) $$\mathrm{d}g = g\mathrm{d}f^c g - \mathrm{d}f.$$

The integrability condition for (2.7) is easily checked[16] to be the isothermic condition

$$\mathrm{d}f \wedge \mathrm{d}f^c = 0$$

---

[16]We shall see an illuminating proof below on page 34.

so that, for any initial condition, we may locally solve (2.7) for g and then defining $\hat{f}$ by $\hat{f} = f + g$, we have

$$g^{-1} \mathrm{d}\hat{f} g^{-1} = r \mathrm{d} f^c$$

so that $\hat{f}$ is a Darboux transform of f.

Notation. For future use, fix a base-point $o \in M$ and let $f : (M, Q) \to \mathbb{R}^n$ be isothermic with Christoffel transform f^c. We denote by $\mathcal{D}_r^v f$ the Darboux transform $\hat{f} = f + g$ where g solves (2.7) with $\hat{f}(o) = v$.

If we do not wish to emphasise the initial condition, we shall simply write $\mathcal{D}_r f$.

So let $\mathcal{D}_r f = \hat{f} = f + g$ be a Darboux transform of $f : (M, Q) \to \mathbb{R}^n$. The demand that $Q_{\hat{f}} = Q$ fixes the Christoffel transform of $\hat{f}$ so that

(2.8) $$d\hat{f}^c = r^{-1} g^{-1} \mathrm{d} f g^{-1}.$$

On the other hand, a well-known symmetry of Riccati equations tells us that g^{-1} must solve a Riccati equation also: indeed,

$$\begin{aligned}
\mathrm{d}(rg)^{-1} &= -r^{-1} g^{-1} \mathrm{d} g g^{-1} \\
&= r^{-1} g^{-1} \mathrm{d} f g^{-1} - \mathrm{d} f^c \\
&= r(rg)^{-1} \mathrm{d} f (rg)^{-1} - \mathrm{d} f^c.
\end{aligned}$$

Thus $(rg)^{-1}$ solves the r-Riccati equation for f^c so that $\widehat{f^c} = f^c + (rg)^{-1} = \mathcal{D}_r f^c$ with

(2.9) $$\mathrm{d}\widehat{f^c} = r^{-1} g^{-1} \mathrm{d} f g^{-1}.$$

Comparing equations (2.8) and (2.9), we see that $\widehat{f^c} = \hat{f}^c$ up to a translation and we have proved a theorem due to Bianchi [2] when $n = 3$ and Hertrich-Jeromin–Pedit [44] when $n = 4$:

Theorem 2.8. *Christoffel and Darboux transforms commute.*

Thus, once we have fixed the Christoffel transform of f, we have a *unique* Christoffel transform of any $\mathcal{D}_r f$ with all ambiguity of scaling and translation removed. Otherwise said, we may think of the Darboux transform as a transform of *Christoffel pairs* $(f, f^c) \mapsto (f + g, f^c + (rg)^{-1})$.

It is humbling to discover that the geometrical description of this construction of the Darboux transform of f^c was already known to Bianchi even though he did not have the Riccati equation: let $P, P_1, \bar{P}, \bar{P}_1$ denote corresponding points on $f, \hat{f}, f^c, \widehat{f^c}$, he writes [2, p. 105]:

> I segmenti PP_1, $\bar{P}\bar{P}_1$ sono paralleli ed il prodotto delle loro lunghezze è constante $= 2/m$.

(Our r is Bianchi's $m/2$.)

Exercise 2.1. Show that if $\hat{f} = \mathcal{D}_r f$ then $f = \mathcal{D}_r^{f(o)} \hat{f}$. Thus $f = \mathcal{D}_r \mathcal{D}_r f$.

To justify our terminology and make contact with the classical literature, we turn to the geometry of our constructions. So let $(f, \hat{f})$ be a Darboux pair. In view of (2.6) and Lemma 1.6, we see that

$$\mathrm{Im}\, \mathrm{d} f = \mathrm{Im}\, g \mathrm{d}\hat{f} g^{-1}$$

so that Proposition 1.7 tells us that f and $\hat{f}$ are enveloping surfaces of a 2-sphere congruence S. We have already seen that f and $\hat{f}$ induce the same conformal structure on M: in classical terms, this means that S is a *conformal sphere congruence.* Again, we know that $Q_f = Q_{\hat{f}}$ so that f and $\hat{f}$ have the same curvature lines. This condition was also well known in the classical literature: recall that a 2-sphere congruence S induces a parallel isomorphism of the normal bundles of its enveloping surfaces f and $\hat{f}$ via $N \mapsto gNg^{-1}$. The congruence is *Ribaucour* if corresponding normals have the same principal directions, that is, if the shape operators A^N and $\hat{A}^{gNg^{-1}}$ of f and $\hat{f}$ commute for each normal N to f.

We therefore conclude: *a Darboux pair consists of the enveloping surfaces of a conformal Ribaucour congruence of 2-spheres.*

If we exclude degenerate cases, the converse is also true: recall that a sphere congruence $S : M \to G^+_{n-2}(\mathbb{R}^{n+1,1})$ is *full* if its image is not contained in some fixed hyperplane.

Exercise 2.2. Suppose that $S : M \to G^+_{n-2}(\mathbb{R}^{n+1,1})$ is not full and has an enveloping surface $f : M \to \text{Ł}$. Reflect f in the fixed hyperplane[17] to get a second enveloping surface $\hat{f}$ and so conclude that S is conformal and Ribaucour.

We now have:

Proposition 2.9. *Let $f, \hat{f}$ envelope a full conformal Ribaucour 2-sphere congruence S and suppose f and $\hat{f}$ have no umbilic points in common. Then $(f, \hat{f})$ is a Darboux pair.*

Proof. Let N be normal along f so that gNg^{-1} is normal along $\hat{f}$. The second fundamental form $\hat{b}^{gNg^{-1}}$ of $\hat{f}$ along gNg^{-1} is given by

$$\hat{b}^{gNg^{-1}}_{U,V} = -\big(\mathrm{d}_U(gNg^{-1}), \mathrm{d}_V \hat{f}\big)$$

and we know from Exercise 1.28 that

$$\mathrm{d}(gNg^{-1}) = g\big(\mathrm{d}N - 2(g^{-1}, N)g^{-1}\mathrm{d}\hat{f}g - 2(g^{-1}, N)\mathrm{d}f\big)g^{-1}$$

whence

$$\hat{b}^{gNg^{-1}}_{U,V} = \Big(g\big(\mathrm{d}_U N - 2(g^{-1}, N)\mathrm{d}_U f\big)g^{-1}, \mathrm{d}\hat{f}_V\Big) - 2(g^{-1}, N)\big(\mathrm{d}\hat{f}_U, \mathrm{d}\hat{f}_V\big).$$

It is not difficult to check that if $(\mathrm{d}N, \mathrm{d}f) = 2(g^{-1}, N)(\mathrm{d}f, \mathrm{d}f)$ at some point then the same identity is also true for $\hat{f}$ at that point:

$$(\mathrm{d}(gNg^{-1}), \mathrm{d}\hat{f}) = -2(g^{-1}, gNg^{-1})(\mathrm{d}\hat{f}, \mathrm{d}\hat{f}).$$

Thus our exclusion of common umbilics prevents this possibility occurring for all N. So choose N and principal vectors X, Y, orthogonal with respect to f, such that the tangential component of $\mathrm{d}_X N - 2(g^{-1}, N)\mathrm{d}_X f$ is a non-zero multiple of $d_X f$. Since S is conformal and Ribaucour, X, Y are orthogonal for $\hat{f}$ and principal for gNg^{-1} so that

$$0 = \big(g(\mathrm{d}_X N - 2(g^{-1}, N)\mathrm{d}_X f)g^{-1}, \mathrm{d}_Y \hat{f}\big)$$

whence

$$0 = (d_X f, g^{-1}\mathrm{d}_Y \hat{f}g).$$

---

[17]If this hyperplane has degenerate metric, take the second surface to be constant.

We therefore conclude that there are are functions μ_1, μ_2 such that

$$\mathrm{d}_X f = \mu_1 g^{-1} \mathrm{d}_X \hat{f} g$$
$$\mathrm{d}_Y f = \mu_2 g^{-1} \mathrm{d}_Y \hat{f} g.$$

Since $(\mathrm{d}f, \mathrm{d}f)$ and $(\mathrm{d}\hat{f}, \mathrm{d}\hat{f})$ are conformally equivalent, we get $\mu_1^2 = \mu_2^2$ and there are only two possibilities: either $\mu_1 = -\mu_2$ which quickly gives

$$\mathrm{d}f \wedge g^{-1} \mathrm{d}\hat{f} g = 0$$

so that $(f, \hat{f})$ are a Darboux pair, or,

$$\mathrm{d}f = \mu_1 g^{-1} \mathrm{d}\hat{f} g$$

which, by Proposition 1.8, forces S to be non-full. $\qquad\qquad\square$

Thus, modulo umbilics, a Darboux pair is exactly a pair of enveloping surfaces of a full conformal Ribaucour 2-sphere congruence and, for $n = 3$, it is this latter formulation that Darboux gave [27].

Darboux's own construction of the Darboux transforms of a given isothermic surface in $\mathbb{R}^3$ proceeded by solving a linear differential system in $\mathbb{R}^{4,1}$ with the algebraic constraint that the solution lie in the light cone Ł. It is instructive to compare this approach with ours: for (f, f^c) a Christoffel pair, contemplate the Lie algebra valued 1-form $B \in \Omega^1 \otimes \varnothing$ given by

$$B = \begin{pmatrix} 0 & \mathrm{d}f \\ r\mathrm{d}f^c & 0 \end{pmatrix}.$$

The Maurer–Cartan equations $\mathrm{d}B + \frac{1}{2}[B \wedge B] = 0$ reduce in this case to the isothermic condition $\mathrm{d}f \wedge \mathrm{d}f^c = 0$ so that the linear differential system

(2.10) $$\mathrm{d}\omega + B\omega = 0$$

for $\omega : M \to \mathbb{R}^{n+1,1}$ is integrable (indeed, one integrates the Maurer–Cartan equations to find $F : M \to \varnothing$ with $F^{-1}\mathrm{d}F = B$ and then solutions of (2.10) are given by $\omega = F^{-1}\omega_0$ for constant ω_0). Clearly, (ω, ω) is an integral of (2.10) so that, in particular, Ł is preserved by the integral flows.

The linear system (2.10) with the constraint $(\omega, \omega) = 0$ is, up to gauge, the system considered by Darboux[18].

Now let $\omega : M \to Ł \subset C\ell_n(2)$ be a solution of (2.10) given by

$$\omega = \begin{pmatrix} v & s \\ t & -v \end{pmatrix}$$

and let $g : M \to \mathbb{R}^n \cup \{\infty\}$ be its stereo-projection. Thus $g = v/t$. I claim that g solves our Riccati equation (2.7). Indeed, the action of $\varnothing$ on $\mathbb{R}^{n+1,1} \subset C\ell_n(2)$ is by commutator of Clifford matrices so that (2.10) reads

$$\begin{pmatrix} \mathrm{d}v & \mathrm{d}s \\ \mathrm{d}t & -\mathrm{d}v \end{pmatrix} + \left[\begin{pmatrix} 0 & \mathrm{d}f \\ r\mathrm{d}f^c & 0 \end{pmatrix}, \begin{pmatrix} v & s \\ t & -v \end{pmatrix} \right] = 0$$

from which we get

$$\mathrm{d}v = sr\mathrm{d}f^c - t\mathrm{d}f$$
$$\mathrm{d}t = -r\mathrm{d}f^c v - rv\mathrm{d}f^c$$

---

[18]For a recent account of Darboux's approach see [11].

whence

$$\mathrm{d}g = \frac{1}{t}\mathrm{d}v - \frac{\mathrm{d}t}{t^2}v = \frac{s}{t}r\mathrm{d}f^c - \mathrm{d}f + r\mathrm{d}f^c\frac{v^2}{t^2} + r\frac{v}{t}\mathrm{d}f^c\frac{v}{t}$$
$$= rg\mathrm{d}f^c g - \mathrm{d}f$$

where we have used $v^2 + st = 0$ (since ω is Ł-valued).

We end our present discussion of the Darboux transform by characterising the frames of Darboux pairs. Recall that a frame of a pair of maps $(f, \hat{f})$ is a map $F : M \to \mathrm{SL}(\Gamma_n)$ such that

$$f = F \cdot 0 \qquad \hat{f} = F \cdot \infty.$$

We prove:

Theorem 2.10. *Let F frame $(f, \hat{f})$ with*

$$F^{-1}\mathrm{d}F = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \Omega^1 \otimes \emptyset.$$

Then $(f, \hat{f})$ is a Darboux pair if and only if $\beta \wedge \gamma = 0$.

In this case, if f is isothermic with respect to a polarisation Q, $\hat{f} = \mathcal{D}_r f$ where r is given by

$$(\beta, \gamma)^{2,0} = -rQ.$$

Proof. The first thing to note is that the conditions on β, γ are independent of the choice of frame: if $\hat{F}$ is another frame of $(f, \hat{f})$ then $\hat{F} = Fk$ for $k : M \to \mathrm{SL}(\Gamma_n)$ with $k \cdot 0 = 0$ and $k \cdot \infty = \infty$. Thus

$$k = \begin{pmatrix} a & 0 \\ 0 & a^{-t} \end{pmatrix}$$

for $a : M \to \Gamma_n$, and setting

$$\hat{F}^{-1}\mathrm{d}\hat{F} = \begin{pmatrix} \hat{\alpha} & \hat{\beta} \\ \hat{\gamma} & \hat{\delta} \end{pmatrix}$$

we readily compute that

$$\hat{\beta} = a^{-1}\beta a^{-t}, \qquad \hat{\gamma} = a^t\gamma a.$$

Thus

$$\hat{\beta} \wedge \hat{\gamma} = a^{-1}(\beta \wedge \gamma)a$$
$$(\hat{\beta}, \hat{\gamma}) = -\tfrac{1}{2}(a^{-1}\beta\gamma a + a^t\gamma\beta a^{-t}) = (\beta, \gamma)$$

where the last equality follows since a^t and a^{-1} are collinear and so have the same adjoint action.

Thus we are free to choose a convenient frame to establish the theorem. With $g = \hat{f} - f$, we take

$$F = \begin{pmatrix} \hat{f}g^{-1} & f \\ g^{-1} & 1 \end{pmatrix}$$

so that, by Exercise 1.25,

$$F^{-1}\mathrm{d}F = \begin{pmatrix} (\mathrm{d}f)g^{-1} & \mathrm{d}f \\ -g^{-1}(\mathrm{d}\hat{f})g^{-1} & -g^{-1}\mathrm{d}f \end{pmatrix}.$$

Thus $\beta = \mathrm{d}f$, $\gamma = -g^{-1}(\mathrm{d}\hat{f})g^{-1}$ and the vanishing of $\beta \wedge \gamma$ is precisely the condition that $(f, \hat{f})$ is a Darboux pair of isothermic surfaces.

Moreover, if $f : (M, Q) \to \mathbb{R}^n$ is isothermic with Christoffel transform f^c and $\hat{f} = \mathcal{D}_r f$, we have

$$g^{-1}(\mathrm{d}\hat{f})g^{-1} = r\mathrm{d}f^c$$

so that

$$(\beta, \gamma)^{2,0} = -r(\mathrm{d}f, \mathrm{d}f^c)^{2,0} = -rQ.$$

$\square$

Remark. The geometric content of this result is that a Darboux pair is the same as a *curved flat* in the symmetric space $S^n \times S^n \setminus \Delta$ of pairs of distinct points in S^n. We shall explain this in Section 3.

2.2.3. *The T-transform.* Our final family of transformations, discovered in the classical setting by Calapso [15] and Bianchi [3], have a slightly different flavour: the construction proceeds by solving a Maurer–Cartan equation to build a frame of the new surface. In particular, these new surfaces are only defined up to the action of Möbius group.

We begin with an isothermic surface $f : (M, Q) \to \mathbb{R}^n$, its Christoffel transform f^c and a parameter $r \in \mathbb{R}$. We have already seen that the ø-valued 1-form B_r given by

$$B_r = \begin{pmatrix} 0 & \mathrm{d}f \\ r\mathrm{d}f^c & 0 \end{pmatrix}$$

solves the Maurer–Cartan equations so that, locally, we may integrate to find $F_r : M \to \mathrm{SL}(\Gamma_n)$ with $F_r^{-1}\mathrm{d}F_r = B_r$. Of course, F_r is only determined up to left translation by a constant in $\mathrm{SL}(\Gamma_n)$. Now F_r frames the pair $f_r, \hat{f}_r : M \to \mathbb{R}^n \cup \{\infty\}$ given by

$$f_r = F_r \cdot 0, \qquad \hat{f}_r = F_r \cdot \infty$$

and, since

$$\mathrm{d}f \wedge (r\mathrm{d}f^c) = 0, \qquad (\mathrm{d}f, r\mathrm{d}f^c)^{2,0} = rQ,$$

we immediately deduce from Theorem 2.10:

Theorem 2.11. *For $r \neq 0$, $(f_r, \hat{f}_r)$ are a Darboux pair of isothermic surfaces. Moreover f_r (and so $\hat{f}_r$) are isothermic with respect to (M, Q) and*

$$\hat{f}_r = \mathcal{D}_{-r} f_r.$$

We denote f_r by $\mathcal{T}_r f$ and, following Bianchi, call it a *T-transform* of f. Note that $\mathcal{T}_r f$ is only determined up to the action of $\mathrm{Möb}(n)$.

When $r = 0$ we may take

$$F_0 = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$$

so that $f_0 = f$ and $\hat{f}_0 \equiv \infty$. Thus we take $\mathcal{T}_0 f = f$ modulo $\mathrm{Möb}(n)$.

Our construction seems to depend in an essential way on the frames we obtained by integrating B_r. However, one can use any frame of f as a starting point[19]: indeed, any frame of f is of the form $\widetilde{F}_0 = F_0 P$ where $P : M \to \mathrm{SL}(\Gamma_n)$ has $P \cdot 0 = 0$ and so is of the form

$$P = \begin{pmatrix} p_1 & 0 \\ p_2 & p_3 \end{pmatrix}$$

with $p_1 p_3^t = 1$.

---

[19]I am grateful to Udo Hertrich-Jeromin for explaining this point to me.

Then $\widetilde{F}_r = F_r P$ frames f_r, that is, $F_r P \cdot 0 = F_r \cdot 0 = f_r$. Moreover,

$$\widetilde{F}_r^{-1} d\widetilde{F}_r = P^{-1} B_r P + P^{-1} dP$$

$$= \widetilde{F}_0^{-1} d\widetilde{F}_0 + r P^{-1} \begin{pmatrix} 0 & 0 \\ df^c & 0 \end{pmatrix} P$$

and a short computation using $p_1 p_3^t = 1$ gives:

$$\widetilde{F}_0^{-1} d\widetilde{F}_0 = \begin{pmatrix} * & p_3^t df p_3 \\ * & * \end{pmatrix}$$

$$P^{-1} \begin{pmatrix} 0 & 0 \\ df^c & 0 \end{pmatrix} P = \begin{pmatrix} 0 & 0 \\ p_3^{-1} df^c p_3^{-t} & 0 \end{pmatrix}.$$

The key point now is that $p_3^{-1} df^c p_3^{-t}$ is constructed from $p_3^t df p_3$ in exactly the same way as df^c is constructed from df, that is, via Christoffel's formula (2.3). Indeed,

$$p_3^{-1} \partial f^c p_3^{-t} = \frac{1}{(df, df)} p_3^{-1} (Q \overline{\partial} f) p_3^{-t}$$

$$= \frac{(p_3 p_3^t)^2}{(p_3^t df p_3, p_3^t df p_3)} p_3^{-1} (Q \overline{\partial} f) p_3^{-t}$$

$$= \frac{1}{(p_3^t df p_3, p_3^t df p_3)} Q p_3^t \overline{\partial} f p_3.$$

For $\alpha \in \Omega^1 \otimes \mathbb{R}^n$ a conformal 1-form, that is $(\alpha, \alpha)^{2,0} = 0$, write

$$\alpha = \alpha' + \alpha''$$

with $\alpha' \in \Omega^{1,0} \otimes \mathbb{C}^n$ and $\overline{\alpha'} = \alpha''$ and define $\alpha^c \in \Omega^1 \otimes \mathbb{R}^n$ by

$$\alpha^c = \frac{1}{(\alpha, \alpha)} (Q \alpha'' + \overline{Q} \alpha').$$

Our last calculation now reads

$$(p_3^t df p_3)^c = p_3^{-1} df^c p_3^{-t}$$

and we have proved

Theorem 2.12. *Let $\widetilde{F}$ frame an isothermic surface $f : (M, Q) \to \mathbb{R}^n$ with*

$$\widetilde{F}^{-1} d\widetilde{F} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}.$$

Then

$$\widetilde{B}_r = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} + r \begin{pmatrix} 0 & 0 \\ \beta^c & 0 \end{pmatrix}.$$

solves the Maurer–Cartan equations and if $\widetilde{F}_r^{-1} d\widetilde{F}_r = \widetilde{B}_r$ then $\widetilde{F}_r$ frames $\mathcal{T}_r f$.

As a first application, let us show that, in analogy with the Lie transform of K-surfaces, $\mathcal{T}_r$ gives an action of $\mathbb{R}$ on isothermic surfaces modulo $\text{Möb}(n)$. Indeed, F_r frames f_r with

$$F_r^{-1} dF_r = \begin{pmatrix} 0 & df \\ r df^c & 0 \end{pmatrix}$$

while

$$F_{s+r}^{-1} dF_{s+r} = \begin{pmatrix} 0 & df \\ (s+r) df^c & 0 \end{pmatrix} = F_r^{-1} dF_r + s \begin{pmatrix} 0 & 0 \\ df^c & 0 \end{pmatrix}$$

so that $f_{s+r} = \mathcal{T}_s f_r$ and we have a theorem proved by Hertrich-Jeromin–Musso–Nicolodi [43] for the case $n = 3$.

Theorem 2.13. $\mathcal{T}_{s+r} = \mathcal{T}_s \circ \mathcal{T}_r$ *modulo* $\text{Möb}(n)$.

Again, we can compare the T-transforms of f and f^c: for $r \in \mathbb{R}^\times$, define R_r by

$$R_r = \begin{pmatrix} 0 & \mathrm{sign}(r)/\sqrt{|r|} \\ \sqrt{|r|} & 0 \end{pmatrix}$$

so that $R_r \cdot 0 = \infty$ and $R_r \cdot \infty = 0$.

Exercise 2.3.

$$\mathrm{Ad}\, R_r^{-1} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} \delta & \gamma/r \\ r\beta & \alpha \end{pmatrix}.$$

Then $\hat{F}_r = F_r R_r$ frames $\hat{f}_r = \mathcal{D}_{-r}f$ but, using Exercise 2.3, we have

$$\hat{F}_r^{-1}\mathrm{d}\hat{F}_r = \mathrm{Ad}\, R_r^{-1}(F_r^{-1}\mathrm{d}F_r) = \begin{pmatrix} 0 & \mathrm{d}f^c \\ r\mathrm{d}f & 0 \end{pmatrix}$$

so that $\hat{F}_r$ also frames $\mathcal{T}_r f^c$. We have therefore proved a result due to Bianchi [3] when $n = 3$:

Theorem 2.14. $\mathcal{T}_r f^c = \mathcal{D}_{-r}\mathcal{T}_r f$ *modulo* $\mathrm{M\ddot{o}b}(n)$.

Similarly, we can compute the interaction of Darboux transforms and T-transforms: let $f : (M, Q) \to \mathbb{R}^n$ be isothermic with Christoffel transform f^c and let $\hat{f} = \mathcal{D}_r f$. As usual, frame $(f, \hat{f})$ with

$$F_0 = \begin{pmatrix} \hat{f}g^{-1} & f \\ g^{-1} & 1 \end{pmatrix}$$

so that

$$F_0^{-1}\mathrm{d}F_0 = \begin{pmatrix} (\mathrm{d}f)g^{-1} & \mathrm{d}f \\ -g^{-1}(\mathrm{d}\hat{f})g^{-1} & -g^{-1}\mathrm{d}f \end{pmatrix} = \begin{pmatrix} (\mathrm{d}f)g^{-1} & \mathrm{d}f \\ -r\mathrm{d}f^c & -g^{-1}\mathrm{d}f \end{pmatrix}.$$

Now let F_s frame $(f_s, \hat{f}_s)$ where F_s solves

$$F_s^{-1}\mathrm{d}F_s = F_0^{-1}\mathrm{d}F_0 + s\begin{pmatrix} 0 & 0 \\ \mathrm{d}f^c & 0 \end{pmatrix} = \begin{pmatrix} (\mathrm{d}f)g^{-1} & \mathrm{d}f \\ (s-r)\mathrm{d}f^c & -g^{-1}\mathrm{d}f \end{pmatrix}.$$

Then Theorem 2.12 tells us that $f_s = \mathcal{T}_s f$ while, from Theorem 2.10, we have $\hat{f}_s = \mathcal{D}_{r-s}f_s$.

On the other hand, set $\hat{F}_s = F_s R_{s-r}$ so that $\hat{F}_s$ frames $(\hat{f}_s, f_s)$ and $\hat{F}_0$ frames $(\hat{f}, f)$. Using Exercise 2.3, we have

$$\hat{F}_s^{-1}\mathrm{d}\hat{F}_s = \mathrm{Ad}\, R_{s-r}^{-1}(F_s^{-1}\mathrm{d}F_s) = \begin{pmatrix} -g^{-1}\mathrm{d}f & \mathrm{d}f^c \\ (s-r)\mathrm{d}f & (\mathrm{d}f)g^{-1} \end{pmatrix}$$

$$= \hat{F}_0^{-1}\mathrm{d}\hat{F}_0 + s\begin{pmatrix} 0 & 0 \\ \mathrm{d}f & 0 \end{pmatrix}.$$

Thus, by Theorem 2.12, $\hat{f}_s = \mathcal{T}_s \hat{f}$ and we conclude, as have Hertrich-Jeromin–Musso–Nicolodi when $n = 3$:

Theorem 2.15. $\mathcal{T}_s \mathcal{D}_r f = \mathcal{D}_{r-s}\mathcal{T}_s f$ *modulo* $\mathrm{M\ddot{o}b}(n)$.

2.3. Darboux transforms of generalised H-surfaces.

Recall that a special class of isothermic surfaces is furnished by the generalised H-surfaces. In view of Proposition 2.5, we may characterise these as surfaces f with a unit normal section N such that, for some constant $H \in \mathbb{R}$, $Hf + N$ is dual to f:

$$\mathrm{d}f \wedge (H\mathrm{d}f + \mathrm{d}N) = 0.$$

Fix such an f and seek Darboux transforms of the same kind. For simplicity we choose the polarisation Q so that $f^c = Hf + N$ (when $n = 3$, this amounts to

taking $-Q$ to be the Hopf differential of f). In this case, our Riccati equation has a conserved quantity. Indeed, if g solves

$$\mathrm{d}g = rg\mathrm{d}f^c g - \mathrm{d}f,$$

define $I : M \to \mathbb{R}$ by

$$I = rHg^2 - r\{g, N\} - 1$$

where $\{\,,\,\}$ is the anti-commutator in $C\ell_n$: $\{g, N\} = -2(g, N)$. We compute:

$$\begin{aligned}
\mathrm{d}I &= rH\{g, \mathrm{d}g\} - r\{\mathrm{d}g, N\} - r\{g, \mathrm{d}N\} \\
&= rH\{g, rg\mathrm{d}f^c g - \mathrm{d}f\} - r\{rg\mathrm{d}f^c g - \mathrm{d}f, N\} - r\{g, \mathrm{d}f^c - H\mathrm{d}f\} \\
&= rH\{g, rg\mathrm{d}f^c g\} - r\{rg\mathrm{d}f^c g, N\} - r\{g, \mathrm{d}f^c\}
\end{aligned}$$

where we have used $\{\mathrm{d}f, N\} = 0$ (N is normal to f) and $\mathrm{d}N = \mathrm{d}f^c - H\mathrm{d}f$. Rearranging this last equation and exploiting $\{\mathrm{d}f^c, N\} = 0$ yields

$$\begin{aligned}
\mathrm{d}I &= rHg^2\{rg, \mathrm{d}f^c\} - r\{g, N\}\{rg, \mathrm{d}f^c\} - r\{g, \mathrm{d}f^c\} \\
&= I\{rg, \mathrm{d}f^c\}.
\end{aligned}$$

This is a linear differential equation for I and so, in particular, I vanishes identically if it vanishes at a single point. We therefore conclude:

Lemma 2.16. *If $rHg(o)^2 - r\{g(o), N(o)\} = 1$ then*

(2.11) $$rHg^2 - r\{g, N\} \equiv 1.$$

Exercise 2.4. For *any* Darboux transform $f + g$ of *any* isothermic surface f, show that $\{g, \mathrm{d}f^c\}$ is a closed 1-form.

Now let g satisfy (2.11) and contemplate $\hat{N} = -gNg^{-1}$: a unit normal to $\hat{f} = f + g$. We know that the Christoffel transform $\hat{f}^c$ of $\hat{f}$ is given by

$$\hat{f}^c = f^c + (rg)^{-1} = Hf + N + r^{-1}g^{-1}.$$

On the other hand, (2.11) tells us that $r^{-1} = Hg^2 - \{g, N\}$ and a simple computation gives:

(2.12) $$\hat{f}^c = H(f + g) - gNg^{-1} = H\hat{f} + \hat{N}.$$

Thus $\hat{N}$ is a parallel isoperimetric section for $\hat{f}$ with $(\hat{\mathbf{H}}, \hat{N}) = H$ and we have proved yet another theorem which is due to Bianchi [2] in the classical setting:

Theorem 2.17. *Let f be a generalised H-surface with $(\mathbf{H}, N) = H$ and choose initial condition $v \in \mathbb{R}^n$ and parameter $r \in \mathbb{R}^\times$ so that $g(o) = v - f(o)$ satisfies*

$$rHg(o)^2 - r\{g(o), N(o)\} = 1.$$

Then the Darboux transform $\mathcal{D}_r^v$ is also a generalised H-surface with the same H.

Thus of the $(n + 1)$-dimensional family of Darboux transforms of a generalised H-surface, an n-dimensional family also produce generalised H-surfaces.

When $H \neq 0$, the conserved quantity (2.11) has a simple geometric interpretation: multiplying by H and completing the square gives

$$\frac{H}{r} - 1 \equiv (Hg - N)^2 = \left(H\hat{f} - (Hf + N)\right)^2$$

or, equivalently,

$$\left(\hat{f} - (f + \tfrac{1}{H}N)\right)^2 \equiv \frac{1}{Hr} - \frac{1}{H^2}.$$

Recall that $f + \frac{1}{H}N$ is the parallel generalised H-surface dual to f and conclude that $\hat{f}$ lies on the tube of radius $\sqrt{1/H^2 - 1/Hr}$ about this parallel surface. In particular, we must have

$$\frac{1}{Hr} \leq \frac{1}{H^2}.$$

The extreme case $H = r$ is not without interest: here $\hat{f} = f + \frac{1}{H}N$ so that $\hat{f}$ is simultaneously dual to f and a Darboux transform of f. In fact, this property characterises generalised H-surfaces[20] with $H \neq 0$:

Exercise 2.5. Let $f : (M, Q) \to \mathbb{R}^n$ be isothermic and $H \in \mathbb{R}^\times$. Show that the following are equivalent:

1. f admits a parallel isoperimetric section N with $(\mathbf{H}, N) = H$.
2. f has a Darboux transform which is also dual to f: $\mathcal{D}_r f = rH^{-2} f^c$.
3. f has a unit normal N such that N/H solves a Riccati equation of f.

2.4. **Bianchi permutability and the Clifford algebra cross-ratio.** We begin by stating a permutability theorem for Darboux transforms that was proved by Bianchi [2] when $n = 3$, Hertrich-Jeromin–Pedit [44] when $n = 4$ and, independently of this writer, Schief [62] in full generality:

Bianchi Permutability Theorem. *Let* $f : (M, Q) \to \mathbb{R}^n$ *be isothermic,* $r_1, r_2 \in \mathbb{R}^\times$ *and* $f_1 = \mathcal{D}_{r_1} f$, $f_2 = \mathcal{D}_{r_2} f$ *distinct Darboux transforms of* f. *Then there is a fourth isothermic surface* $\hat{f}$ *such that*

$$\hat{f} = \mathcal{D}_{r_2} f_1 = \mathcal{D}_{r_1} f_2.$$

In these notes, we shall give two proofs of this result using rather different ideas. The first relies on the Clifford algebra cross-ratio to which we now turn:

Definition. Let v_0, v_1, v_2, v_3 be distinct points in $\mathbb{R}^n$. The *Clifford algebra cross-ratio* of these points is given by

$$C(v_0, v_1, v_2, v_3) = (v_1 - v_0)(v_2 - v_1)^{-1}(v_2 - v_3)(v_3 - v_0)^{-1}$$
$$= (v_0 - v_1)(v_1 - v_2)^{-1}(v_2 - v_3)(v_3 - v_0)^{-1} \in C\ell_n.$$

This cross-ratio is almost invariant under the action of the Möbius group:

Exercise 2.6. Let v_0, v_1, v_2, v_3 be distinct points in $\mathbb{R}^n$ and $T \in \mathrm{SL}(\Gamma_n)$ with

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

1. Show that $T \cdot v_1 - T \cdot v_0 = (cv_0 + d)^{-t}(v_1 - v_0)(cv_0 + d)^{-1}$.
   Hint: recall that $a^t d - c^t b = 1$ and that $a^t c \in \mathbb{R}^n$ so that $a^t c = c^t a$.
2. Write $C(v_0, v_1, v_2, v_3) = (v_1 - v_0)(v_1 - v_2)^{-1}(v_3 - v_2)(v_3 - v_0)^{-1}$ and deduce that

$$C(T \cdot v_0, T \cdot v_1, T \cdot v_2, T \cdot v_3) = (cv_0 + d)^{-t} C(v_0, v_1, v_2, v_3)(cv_0 + d)^t.$$

In particular, the condition that four points have real cross-ratio is conformally invariant. In fact, we can say more:

Proposition 2.18 ([22]). $C(v_0, v_1, v_2, v_3) = r \in \mathbb{R}$ *if and only if* v_0, v_1, v_2, v_3 *lie on a circle and have real cross-ratio* r.

---

[20]For CMC surfaces in $\mathbb{R}^3$, this was known to Bianchi [2, footnote p. 132], see also [44].

Proof. Possibly after a Möbius transformation, we may assume that v_0, v_1, v_2, v_3 lie on a $\mathbb{R}^2 \subset \mathbb{R}^n$ so that their cross-ratio lies in $C\ell_2 = \mathbb{H}$. Write $\mathbb{H} = \mathbb{C} \oplus j\mathbb{C}$. Then $\mathbb{R}^2 = j\mathbb{C}$ and, writing $v_i = jz_i$, we see that

$$C(v_0, v_1, v_2, v_3) = jC_{\mathbb{C}}(v_0, v_1, v_2, v_3)j^{-1} = \overline{C_{\mathbb{C}}(v_0, v_1, v_2, v_3)}$$

where $C_{\mathbb{C}}$ is the usual complex cross-ratio which is real if and only if the z_i are concircular[21]. $\qquad\square$

The relevance of the cross-ratio to Bianchi permutability comes from the following considerations: with f isothermic and $f_i = \mathcal{D}_{r_i}f = f + g_i$, $i = 1, 2$, distinct Darboux transforms, suppose that the theorem is true so that we have $\hat{f} = \mathcal{D}_{r_2}f_1 = \mathcal{D}_{r_1}f_2$ and write

$$\hat{f} = f_1 + g_{12} = f_2 + g_{21}.$$

Now

$$d\hat{f} = r_2 g_{12}df_1^c g_{12} = \frac{r_2}{r_1}g_{12}g_1^{-1}dfg_{12}g_1^{-1}$$

and, in the same way, we also have

$$d\hat{f} = \frac{r_1}{r_2}g_{21}g_2^{-1}dfg_{21}g_2^{-1}.$$

Equating these, we arrive at

$$(2.13) \qquad \frac{r_2^2}{r_1^2}dfg_1^{-1}g_{12}g_{21}^{-1}g_2 = g_1g_{12}^{-1}g_{21}g_2^{-1}df.$$

Taking Clifford algebra norms of both sides gives

$$(2.14) \qquad \frac{r_2^2}{r_1^2} = g_1^2 g_{12}^{-2} g_{21}^2 g_2^{-2}$$

and (2.13) becomes

$$(2.15) \qquad [g_1 g_{12}^{-1} g_{21} g_2^{-1}, df] = 0.$$

Now recall that if $f, f + g$ envelope a 2-sphere congruence, $N \mapsto -gNg^{-1}$ is a parallel isomorphism of normal bundles. In the present setting, we therefore arrive at two such isomorphisms between the normal bundles of f and $\hat{f}$ and we make the *ansatz* that these coincide[22]: that is, for N normal to f, we assume,

$$g_{21}g_2Ng_2^{-1}g_{21}^{-1} = g_{12}g_1Ng_1^{-1}g_{12}^{-1}.$$

Rearranging this and multiplying by $g_1^2 g_2^{-2}$ gives us

$$[g_1 g_{12}^{-1} g_{21} g_2^{-1}, N] = 0$$

which taken together with (2.15) tells us that $g_1 g_{12}^{-1} g_{21} g_2^{-1}$ commutes with all of $\mathbb{R}^n$ and so is central in $C\ell_n$. Moreover, using

$$g_1 + g_{12} = \hat{f} - f = g_2 + g_{21}$$

one checks that $g_1 g_{12}^{-1} g_{21} g_2^{-1} \in \mathbb{R}^n \cdot \mathbb{R}^n \subset C\ell_n$. However, when $n > 2$, $\mathbb{R}^n \cdot \mathbb{R}^n$ intersects the centre of $C\ell_n$ in $\mathbb{R}$ alone so we conclude that $g_1 g_{12}^{-1} g_{21} g_2^{-1} \in \mathbb{R}$ and, in view of (2.14), we must have

$$C(f, f_1, \hat{f}, f_2) = g_1 g_{12}^{-1} g_{21} g_2^{-1} = \pm\frac{r_2}{r_1}.$$

---

[21]Indeed, possibly after a second Möbius transformation, we may assume z_0, z_1, z_2 are real and then solve for z_3: $z_3 = (z_2(z - z_0) + rz_0(z_2 - z_1))/(r(z_2 - z_1) + (z_1 - z_0)) \in \mathbb{R}$.

[22]When $n = 3$, this amounts to choosing a sign.

To fix the sign, we consider the degenerate case where $r_1 = r_2$ where, according to Exercise 2.1, we may take $\hat{f} = f$ and then the cross-ratio is $1 = r_2/r_1$. We therefore conclude that we should have

$$(2.16) \qquad\qquad C(f, f_1, \hat{f}, f_2) = \frac{r_2}{r_1}$$

Remark. In the case $n = 4$, Hertrich-Jeromin–Pedit arrive at the quaternionic version of the same *ansatz* by considerations coming from the theory of discrete isothermic surfaces [4, 42].

Exercise 2.7.   1. If (2.16) holds, show that

$$(2.17) \qquad\qquad g_{12} = (g_1 - g_2)r_1 g_2^{-1}(r_2 g_1^{-1} - r_1 g_2^{-1})^{-1}.$$

  2. Deduce from (2.17) that

$$(2.18) \qquad\qquad \hat{f} = (r_2 f_1 g_1^{-1} - r_1 f_2 g_2^{-1})(r_2 g_1^{-1} - r_1 g_2^{-1})^{-1}.$$

To prove our theorem, it remains to show that if $\hat{f}$ is defined by (2.18) then we really do have $\hat{f} = \mathcal{D}_{r_2} f_1 = \mathcal{D}_{r_1} f_2$. To show the first of these amounts to proving that

$$\mathrm{d}g_{12} = r_2 g_{12} \mathrm{d}f_1^c g_{12} - \mathrm{d}f_1,$$

that is,

$$\mathrm{d}g_{12} = \frac{r_2}{r_1} g_{12} g_1^{-1} \mathrm{d}f g_1^{-1} g_{12} - r_1 g_1 \mathrm{d}f^c g_1.$$

This is a tedious but straightforward verification using (2.17).

Exercise 2.8. Check the grisly details!

This completes the proof of the permutability theorem and gives us more. In fact, we have shown (as has Schief [62]):

Theorem 2.19. *Let f be isothermic with distinct Darboux transforms $f_1 = \mathcal{D}_{r_1} f$ and $f_2 = \mathcal{D}_{r_2} f$. Then there is a fourth surface $\hat{f} = \mathcal{D}_{r_2} f_1 = \mathcal{D}_{r_1} f_2$ such that corresponding points on $f, f_1, \hat{f}, f_2$ are concircular with real cross-ratio r_2/r_1.*

We call four surfaces in the configuration of Theorem 2.19 a *Bianchi quadrilateral*.

Our explicit formula for the fourth surface of a Bianchi quadrilateral allows us to give *algebraic* proofs[23] of several results of Bianchi [2] concerning the geometry of such configurations which immediately extend to our n-dimensional setting.

First, let us consider the Christoffel transform of a Bianchi quadrilateral: let $(f, f_1, \hat{f}, f_2)$ be such a quadrilateral and contemplate the Christoffel transforms f^c, $f_1^c = \mathcal{D}_{r_1} f^c = f^c + (r_1 g_1)^{-1}$, $f_2^c = \mathcal{D}_{r_2} f^c = f^c + (r_2 g_2)^{-1}$. We now have three rival Christoffel transforms of $\hat{f}$: $f_1^c + (r_2 g_{12})^{-1}$, $f_2^c + (r_1 g_{21})^{-1}$ and $\hat{f}^c$ given by the permutability theorem so as to make $(f^c, f_1^c, \hat{f}^c, f_2^c)$ a Bianchi quadrilateral[24]. Of course, these three possibilities can only differ by constants but, in fact, they coincide exactly:

Exercise 2.9. If $g_1, g_2, g_{12} \in \mathbb{R}^n$ are given by (2.17) then $g_1^c = (r_1 g_1)^{-1}$, $g_2^c = (r_2 g_2)^{-1}$, $g_{12}^c = (r_2 g_{12})^{-1}$ also satisfy (2.17):

$$g_{12}^c = (g_1^c - g_2^c)r_1(g_2^c)^{-1}(r_2(g_1^c)^{-1} - r_1(g_2^c)^{-1})^{-1}.$$

---

[23]All the material in the remainder of this section resulted from conversations with Udo Hertrich-Jeromin.

[24]That $\hat{f}^c$ is also a Christoffel transform follows from Theorem 2.8.

Thus $\hat{f}^c = f_1^c + (r_2 g_{12})^{-1}$ and, by symmetry, $\hat{f}^c = f_2^c + (r_1 g_{21})^{-1}$. To summarise:

Theorem 2.20. *The Christoffel transform of a Bianchi quadrilateral is also a Bianchi quadrilateral.*

A similar but slightly more elaborate analysis shows that a Darboux transform of a Bianchi quadrilateral is another Bianchi quadrilateral. For this we need a version of the hexahedron lemma of [42]:

Lemma 2.21. *Let $v, v_1, \hat{v}, v_2$ be distinct concircular points in $\mathbb{R}^n$ with Clifford algebra cross-ratio $C(v, v_1, \hat{v}, v_2) = r_2/r_1$ and let $v' \in \mathbb{R}^n$ distinct from v, v_1, v_2. Then, for $r_3 \in \mathbb{R}^\times$, there are unique points $v_1', \hat{v}', v_2'$ such that*

$$C(v, v_1, \hat{v}, v_2) = C(v', v_1', \hat{v}', v_2') = r_2/r_1$$
$$C(v, v', v_1', v_1) = C(v_2, v_2', \hat{v}', \hat{v}) = r_1/r_3$$
$$C(v, v', v_2', v_2) = C(v_1, v_1', \hat{v}', \hat{v}) = r_2/r_3.$$

Moreover, all 8 points lies on a single 2-sphere or plane in $\mathbb{R}^n$.

Proof. The points $v, v_1, \hat{v}, v_2, v'$ lie on a 2-sphere or plane and so, after a Möbius transformation, may be taken to lie on a copy of $\mathbb{R}^2$ where, as in the proof of Proposition 2.18, all Clifford algebra cross-ratios reduce to complex cross-ratios. One now solves

$$C_{\mathbb{C}}\,(v, v', v_1', v_1) = r_1/r_3$$
$$C_{\mathbb{C}}\,(v, v', v_2', v_2) = r_2/r_3$$
$$C_{\mathbb{C}}\,(v', v_1', \hat{v}', v_2') = r_2/r_1$$

to obtain, in turn, $v_1', v_2', \hat{v}' \in \mathbb{C}$ and then checks that the remaining two equations hold: a task best left to a computer algebra engine (c.f. [42]). $\square$

Now suppose that we start with a Bianchi quadrilateral $(f, f_1, \hat{f}, f_2)$ with $f_1 = \mathcal{D}_{r_1} f$, $f_2 = \mathcal{D}_{r_2} f$ and take a third Darboux transform $f' = \mathcal{D}_{r_3} f$ of f. The permutability theorem yields isothermic surfaces

$$f_1' = \mathcal{D}_{r_3} f_1 = \mathcal{D}_{r_1} f'$$
$$f_2' = \mathcal{D}_{r_3} f_2 = \mathcal{D}_{r_2} f'$$

and, finally, thanks to Lemma 2.21,

$$\hat{f}' = \mathcal{D}_{r_3} f' = \mathcal{D}_{r_1} f_2' = \mathcal{D}_{r_2} f_1'.$$

Thus these 8 surfaces form the vertices of a cube all of whose faces are Bianchi quadrilaterals! In particular, we have:

Theorem 2.22. *For suitably chosen initial conditions, the Darboux transform of a Bianchi quadrilateral is a Bianchi quadrilateral.*

As a last application of these ideas, let us show that if the first three surfaces in a Bianchi quadrilateral are generalised H-surfaces with the same $H \neq 0$ then so is the fourth. We begin by examining a degenerate case: so let f be a generalised H-surface with $H \neq 0$ and $f^c = Hf + N$. We have seen that the parallel surface f^c/H is a Darboux transform of f: $f^c/H = \mathcal{D}_H f$. Now take a second Darboux transform $f_1 = \mathcal{D}_{r_1} f$ which is also a generalised H-surface and contemplate the Bianchi quadrilateral $(f, f_1, \hat{f}, f^c/H)$.

Proposition 2.23. $\hat{f} = f_1^c/H.$

Proof. We must check that $C(f, f_1, f_1^c/H, f^c/H) = H/r_1$. However, from (2.11), we know that

$$f_1^c = f^c + (r_1 g_1)^{-1} = Hf_1 - g_1 N g_1^{-1}$$

whence

$$g_{12} = f_1^c/H - f_1 = -g_1 N g_1^{-1}$$
$$g_{21} = f_1^c/H - f^c/H = (r_1 g_1)^{-1}.$$

Finally, $g_2 = N/H$ so that

$$C(f, f_1, f_1^c/H, f^c/H) = -g_1(g_1 N g_1^{-1})^{-1}(r_1 g_1)^{-1}(N/H)^{-1} = H/r_1$$

since $N^2 = -1$. $\square$

Thus a Darboux pair of generalised H-surfaces, together with their parallel H-surfaces form a Bianchi quadrilateral.

We are now in a position to prove:

Theorem 2.24. *Let $(f, f_1, \hat{f}, f_2)$ be a Bianchi quadrilateral with f, f_1, f_2 generalised H-surfaces with the same $H \neq 0$. Then $\hat{f}$ is also a generalised H surface.*

Proof. Consider the configuration of 8 surfaces obtained from Lemma 2.21 starting with $(f, f_1, \hat{f}, f_2)$ and $f' = f^c/H$. Proposition 2.23 tells us that $f_1' = f_1^c$ and $f_2' = f_2^c$ while, from Theorem 2.20, we have

$$C(f^c, f_1^c, \hat{f}^c, f_2^c) = r_2/r_1.$$

Now, an obvious scaling symmetry of the cross-ratio gives

$$C(f^c, f_1^c, \hat{f}^c, f_2^c) = C(f^c/H, f_1^c/H, \hat{f}^c/H, f_2^c/H)$$

so that

$$C(f^c/H, f_1^c/H, \hat{f}^c/H, f_2^c/H) = r_2/r_1 = C(f^c/H, f_1^c/H, \hat{f}', f_2^c/H).$$

We conclude that $\hat{f}' = \hat{f}^c/H$, that is, $\hat{f}^c/H = \mathcal{D}_H \hat{f}$ so that, by Exercise 2.5, $\hat{f}$ is a generalised H-surface also. $\square$

2.5. **Isothermic surfaces via the vector Calapso equation.** Let us pause from our main development and digress[25] to consider the approach of Calapso [15, 16] to isothermic surfaces. For $n = 3$, he reduced the problem to the study of a fourth-order non-linear partial differential equation for a function that turns out to be (the coefficient of) the conformal Hopf differential in CCL coordinates. This PDE is equivalent to the stationary version of the second flow of the Davey–Stewartson II hierarchy [37] —a hierarchy of integrable PDE with mysterious[26] (to this author) connections to conformal geometry [51, 52].

In this section, we describe a simple generalisation of Calapso's approach which treats isothermic surfaces in $\mathbb{R}^n$ and was also arrived at independently by Schief [62]. For this we adapt an argument of [11] and so temporarily abandon our Clifford algebra formalism to work with frames in $O^+(n+1, 1)$.

Fix a basis $e_0, \ldots, e_{n+1}$ of $\mathbb{R}^{n+1,1}$ with $e_1, \ldots, e_n$ space-like orthogonal and $e_0, e_{n+1} \in L^+$ with $(e_0, e_{n+1}) = -\frac{1}{2}$. A map $F : M \to O^+(n+1, 1)$ frames an immersion $\langle f \rangle : M \to \mathbb{P}(L)$ if $\pi(Fe_0) = \langle f \rangle$, that is,

$$Fe_0 \in \langle f \rangle.$$

---

[25]This section may be omitted from a first reading.

[26]*Note added in December 2001*: these matters are now a little less mysterious to me, see [13]

Let $\langle f \rangle : M \to \mathbb{P}(\mathrm{L})$ be isothermic and fix $z = x + iy$ a CCL coordinate. We are going to construct an essentially unique and Möbius invariant frame for $\langle f \rangle$. Firstly, choose $f : M \to \mathrm{L}^+$ to be the (unique) lift of $\langle f \rangle$ with

$$(\mathrm{d}f, \mathrm{d}f) = \mathrm{d}x^2 + \mathrm{d}y^2$$

and set $X = f_x$, $Y = f_y$: these are orthonormal and space-like. Now contemplate the conformal Gauss map of $\langle f \rangle$ (cf page 15):

$$Z_{\langle f \rangle} = \langle f, f_x, f_y, f_{xx} + f_{yy} \rangle^\perp$$

which is isomorphic to the normal bundle $\mathcal{N}_{\langle f \rangle}$ and so a flat bundle with respect to its induced connection. Choose orthonormal parallel sections $N_1, \ldots, N_{n-2}$ of $Z_{\langle f \rangle}$. Finally, let $\hat{f} : M \to \mathrm{L}^+$ be (uniquely) determined by the demands that $\hat{f}$ is orthogonal to $X, Y, N_1, \ldots, N_{n-2}$ and that $(f, \hat{f}) = -\frac{1}{2}$.

This data defines a frame $F : M \to \mathrm{O}^+(n+1, 1)$ of $\langle f \rangle$ such that

$$Fe_0 = f$$
$$Fe_1 = X, \quad Fe_2 = Y$$
$$Fe_i = N_{i-2} \quad \text{for } 3 \le i \le n$$
$$Fe_{n+1} = \hat{f}$$

which is completely determined by $\langle f \rangle$ and z up to the right action of $\varnothing[n-2]$ permuting the choice of parallel framing of $Z_{\langle f \rangle}$.

Each N_i is parallel so that $\mathrm{d}N_i \in \langle f, f_x, f_y \rangle$. Moreover, x, y are curvature line coordinates so there are functions $\kappa_i^{(1)}, \kappa_i^{(2)}$ such that

$$\mathrm{d}N_i = -\kappa_i^{(1)} f_x \mathrm{d}x - \kappa_i^{(2)} f_y \mathrm{d}y + \tau_i f$$

for some 1-form τ_i. Now $(N_i, f_{xx} + f_{yy}) = 0$ while

$$\kappa_i^{(1)} = -(N_{i,x}, f_x) = (N_i, f_{xx})$$
$$\kappa_i^{(2)} = (N_i, f_{yy})$$

so that

$$\kappa_i^{(1)} + \kappa_i^{(2)} = 0.$$

We therefore set $\kappa_i = \kappa_i^{(1)}$ and conclude

(2.19) $$\mathrm{d}N_i = -\kappa_i X \mathrm{d}x + \kappa_i Y \mathrm{d}y + \tau_i f.$$

The κ_i are the components of the conformal Hopf differential with respect to the frame $N_1, \ldots, N_{n-2}$ of $Z_{\langle f \rangle}$ and our CCL coordinate $z = x + iy$:

Exercise 2.10. Recall the definition of the conformal Hopf differential from page 14. Show that

$$K_{\langle f \rangle}(N_i + \langle f \rangle) = \kappa_i$$

Remark. If, instead of the isometric lift, we take a Euclidean lift $f' : M \to E_{v_\infty} \subset \mathrm{L}^+$, we can use the Euclidean normal bundle and parallel sections $N_1', \ldots, N_{n-2}'$ to compute $K_{\langle f \rangle}$. We then get

$$K_{\langle f \rangle}(N_i' + \langle f \rangle) = \frac{e^u}{2}(\kappa_i' - \kappa_i'')$$

where $(\mathrm{d}f', \mathrm{d}f') = e^{2u}(\mathrm{d}x^2 + \mathrm{d}y^2)$ and the κ_i', κ_i'' are the Euclidean principal curvatures for N_i'. This gives the formulation of Calapso [15] and Schief [62].

Returning to our frame, we note that

$$\mathrm{d}X, \mathrm{d}Y \perp \langle X, Y \rangle$$

since X, Y are an orthonormal coordinate frame for a flat metric on M and, taking this together with (2.19), we compute the Maurer–Cartan form of F:

$$B = F^{-1}\mathrm{d}F = \begin{pmatrix} & \chi_1 & \chi_2 & \tau & \\ \mathrm{d}x & & & -\kappa\mathrm{d}x & -\chi_1 \\ \mathrm{d}y & & & \kappa\mathrm{d}y & -\chi_2 \\ & \kappa^T\mathrm{d}x & -\kappa^T\mathrm{d}y & & -\tau^T \\ & -\mathrm{d}x & -\mathrm{d}y & & \end{pmatrix}$$

where $\kappa = (\kappa_1, \ldots, \kappa_{n-2})$, $\tau = (\tau_1, \ldots, \tau_{n-2})$ and χ_1, χ_2 are two more 1-forms.

Now B satisfies the Maurer–Cartan equations. Conversely, any B of the above form that satisfies the Maurer–Cartan equations can be locally integrated to give $F : M \to \mathrm{O}^+(n+1,1)$ with $B = F^{-1}\mathrm{d}F$. If we then define $f = Fe_0$, $N_i = Fe_{i+2}$, $1 \le i \le n-2$, we see that

$$f_x = Fe_1 \qquad f_y = Fe_2$$

so that the N_i are normal to f. Moreover, we have

$$\mathrm{d}N_i = -\kappa_i f_x \mathrm{d}x + \kappa_i f_y \mathrm{d}y + \tau_i f$$

which shows that x, y are CCL coordinates so that $\langle f \rangle$ is isothermic and, in addition, that the N_i are a parallel frame for the conformal Gauss map of $\langle f \rangle$.

So let us examine the Maurer–Cartan equations of B: these amount to

(2.20a) $$\chi_1 \wedge \mathrm{d}x + \chi_2 \wedge \mathrm{d}y = 0$$

(2.20b) $$\chi_2 \wedge \mathrm{d}x - \chi_1 \wedge \mathrm{d}y + (\kappa, \kappa)\mathrm{d}y \wedge \mathrm{d}x = 0$$

(2.20c) $$\mathrm{d}(\kappa\mathrm{d}x) + \tau \wedge \mathrm{d}x = 0$$

(2.20d) $$\mathrm{d}(\kappa\mathrm{d}y) - \tau \wedge \mathrm{d}y = 0$$

(2.20e) $$\mathrm{d}\tau - \chi_1 \wedge \kappa\mathrm{d}x + \chi_2 \wedge \kappa\mathrm{d}y = 0$$

(2.20f) $$\mathrm{d}\chi_1 + \tau \wedge \kappa\mathrm{d}x = 0$$

(2.20g) $$\mathrm{d}\chi_2 - \tau \wedge \kappa\mathrm{d}y$$

where we have written (κ, κ) for $\sum_{i=1}^{n-2} \kappa_i^2$.

Write

$$\chi_i = \chi_{i1}\mathrm{d}x + \chi_{i2}\mathrm{d}y.$$

Then (2.20a) is equivalent to $\chi_{12} = \chi_{21}$ and we denote this common value by ψ.

Similarly, (2.20b) is equivalent to

(2.21) $$\chi_{11} + \chi_{22} = -(\kappa, \kappa)$$

so we write

(2.22) $$\chi_{11} = \tfrac{1}{2}\big(u - (\kappa, \kappa)\big), \quad \chi_{22} = \tfrac{1}{2}\big(-u - (\kappa, \kappa)\big)$$

for some function $u : M \to \mathbb{R}$.

The vector valued equations (2.20c) and (2.20d) amount to

(2.23) $$\tau = \kappa_x\mathrm{d}x - \kappa_y\mathrm{d}y$$

while (2.20e) gives

$$\mathrm{d}\tau = 2\psi\kappa\mathrm{d}y \wedge \mathrm{d}x$$

or, using (2.23),

$$\kappa_{xy} = \psi\kappa.$$

Finally, (2.20f) and (2.20g) give

(2.24a)
$$\tfrac{1}{2}u_y = \psi_x + (\kappa, \kappa)_y$$

(2.24b)
$$\tfrac{1}{2}u_x = -\psi_y - (\kappa, \kappa)_x.$$

Now $du = 0$ which is the same as

$$\Delta\psi + 2(\kappa, \kappa)_{xy} = 0.$$

Thus the Maurer–Cartan equations for B boil down to the *vector Calapso equation*:

(2.25a)
$$\kappa_{xy} = \psi\kappa$$

(2.25b)
$$\Delta\psi + 2(\kappa, \kappa)_{xy} = 0.$$

Remark. When $n = 3$, κ is scalar and we can eliminate ψ to obtain Calapso's original equation[27]:

$$\Delta\left(\frac{\kappa_{xy}}{\kappa}\right) + 2(\kappa^2)_{xy} = 0.$$

Conversely, given a solution κ, ψ of the vector Calapso equation (2.25), we integrate (2.24) to obtain u, define τ by (2.23) and finally χ_i by (2.22) together with $\chi_{12} = \chi_{21} = \psi$ to get a Maurer–Cartan solution and so a frame of an isothermic surface, unique up to a Möbius transformation.

In fact, we get more from this analysis: there is a constant of integration in the definition of u. Replacing u by $u + r$ gives us a new Maurer–Cartan solution

$$B_{r/2} = B + \frac{r}{2}\begin{pmatrix} dx & -dy & \\ & & -dx \\ & & dy \\ & & \end{pmatrix}$$

and so a new isothermic surface $\langle f\rangle_{r/2}$.

We have seen this before. In our Clifford algebra formulation,

$$B = \begin{pmatrix} * & e_1 dx + e_2 dy \\ e_1\chi_1 + e_2\chi_2 & * \end{pmatrix}$$

and

$$B_{r/2} = B + \frac{r}{2}\begin{pmatrix} 0 & 0 \\ e_1 dx - e_2 dy & 0 \end{pmatrix}.$$

One easily checks that

$$(e_1 dx + e_2 dy)^c = e_1 dx - e_2 dy$$

so that, by Theorem 2.12, $\langle f\rangle_{r/2}$ is the T-transform $\mathcal{T}_{r/2}\langle f\rangle$ of $\langle f\rangle$.

To summarise: *each solution of the vector Calapso equation (2.25) gives rise to the 1-parameter family of T-transforms of an isothermic surface and conversely.*

---

[27]In fact, this equation first appeared in the thesis of Rothe [60].

3. Curved flats

The rich transformation theory of isothermic surfaces strongly suggests the presence of an underlying integrable system. This is indeed the case: the integrable system in question is that of *curved flats* discovered by Ferus–Pedit [38] which is very closely related to the "n-dimensional system" of Terng [65].

It is a main result of [11] that Darboux pairs in $\mathbb{R}^3$ amount to curved flats in a certain Grassmannian. In this section, we shall show that such a result holds in arbitrary codimension and, in so doing, unify much of the transformation theory of Section 2.

3.1. **Curved flats in symmetric spaces.** Let G/K be a symmetric space. Thus G is a Lie group (usually, for us, semisimple) with an involution $\tau : G \to G$ and K is a closed subgroup open in the fixed set of τ. The derivative at 1 of τ is an involution, also called τ, of the Lie algebra $\mathfrak{g}$ of G. We have a decomposition

$$(3.1) \qquad\qquad \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

into ± 1-eigenspaces of τ. The $+1$-eigenspace $\mathfrak{k}$ is the Lie algebra of K and, since τ is an involution of $\mathfrak{g}$, we have:

$$(3.2) \qquad\qquad [\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}, \qquad [\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}, \qquad [\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}.$$

The left action of G on G/K differentiates to give a surjection $\mathfrak{g} \to T_{gK} G/K$:

$$\xi \mapsto \left. \frac{\mathrm{d}}{\mathrm{d}t} \right|_{t=0} (\exp t\xi)gK$$

with kernel $\mathrm{Ad}(g)\mathfrak{k}$ which therefore restricts to give an isomorphism $\mathrm{Ad}(g)\mathfrak{p} \cong T_{gk} G/K$. In this way, we view each tangent space to G/K as a subspace of $\mathfrak{g}$.

Definition ([38])**.** An immersion $\phi : M \to G/K$ of a manifold M is a *curved flat* if each $\mathrm{d}\phi(T_p M)$ is an abelian subalgebra of $\mathfrak{g}$ (where $T_{\phi(p)} G/K \subset \mathfrak{g}$ as above).

Under mild conditions on G, this amounts to the demand that the curvature operator of the canonical connection of G/K vanishes on each $\bigwedge^2 \mathrm{d}\phi(T_p M)$.

A *frame* of ϕ is a map $F : M \to G$ which is mapped onto ϕ by the coset projection $G \to G/K$:

$$\phi = FK.$$

Since the coset projection is locally trivial, frames exist locally and if F is one such, any other is of the form Fk with $k : M \to K$.

As we have already seen, a map $F : M \to G$ is determined by its Maurer–Cartan form $A = F^{-1}\mathrm{d}F \in \Omega^1 \otimes \mathfrak{g}$ which satisfies the Maurer–Cartan equations:

$$(3.3) \qquad\qquad \mathrm{d}A + \tfrac{1}{2}[A \wedge A] = 0$$

where

$$[A \wedge B]_{X,Y} = [A_X, B_Y] - [A_Y, B_X].$$

Conversely, if $A \in \Omega^1 \otimes \mathfrak{g}$ solves (3.3) then we can locally integrate to find $F : M \to G$, unique up to left multiplication by constants, with $A = F^{-1}\mathrm{d}F$.

For F a frame of ϕ and $A = F^{-1}\mathrm{d}F$, write

$$A = A_{\mathfrak{k}} + A_{\mathfrak{p}}$$

according to the decomposition (3.1). Viewing $\mathrm{d}\phi$ as a $\mathfrak{g}$-valued 1-form, we have

$$\mathrm{d}\phi = \mathrm{Ad}(F)A_{\mathfrak{p}}$$

so that ϕ is a curved flat if and only if

$$[A_{\mathfrak{p}} \wedge A_{\mathfrak{p}}] = 0.$$

Now the Maurer–Cartan equations (3.3) decompose into their components in $\mathfrak{k}$ and $\mathfrak{p}$ which, in view of (3.2), read

$$\mathrm{d}A_k + \tfrac{1}{2}[A_{\mathfrak{k}}, A_{\mathfrak{k}}] + \tfrac{1}{2}[A_{\mathfrak{p}} \wedge A_{\mathfrak{p}}] = 0$$
$$\mathrm{d}A_{\mathfrak{p}} + [A_{\mathfrak{k}} \wedge A_{\mathfrak{p}}] = 0$$

so that ϕ is a curved flat if and only if these equations decouple further to give:

(3.4a) $$\mathrm{d}A_k + \tfrac{1}{2}[A_{\mathfrak{k}}, A_{\mathfrak{k}}] = 0$$

(3.4b) $$\mathrm{d}A_{\mathfrak{p}} + [A_{\mathfrak{k}} \wedge A_{\mathfrak{p}}] = 0$$

(3.4c) $$[A_{\mathfrak{p}} \wedge A_{\mathfrak{p}}] = 0$$

Now observe that (3.4) are the coefficients of a *spectral parameter* $\lambda \in \mathbb{R}$ in the Maurer–Cartan equations for the pencil of 1-forms $A_\lambda \in \Omega^1 \otimes \mathfrak{g}$ given by

$$A_\lambda = A_{\mathfrak{k}} + \lambda A_{\mathfrak{p}}.$$

That is,

Proposition 3.1. *Let* $F : M \to G$ *with* $F^{-1}\mathrm{d}F = A_{\mathfrak{k}} + A_{\mathfrak{p}}$. *Then* F *frames a curved flat if and only if* $A_\lambda = A_{\mathfrak{k}} + \lambda A_{\mathfrak{p}}$ *satisfies*

$$\mathrm{d}A_\lambda + \tfrac{1}{2}[A_\lambda \wedge A_\lambda] = 0$$

for all $\lambda \in \mathbb{R}$.

We have therefore arrived at a *zero curvature formulation* of the curved flat condition.

As an immediate consequence, we see that curved flats come in 1-parameter families: for each $\lambda \in \mathbb{R}$, we can locally integrate to find $F_\lambda : M \to G$ with $F_\lambda^{-1}\mathrm{d}F_\lambda = A_\lambda$ and, since each $(A_\lambda)_{\mathfrak{p}} = \lambda A_{\mathfrak{p}}$, we have

$$[(A_\lambda)_{\mathfrak{p}} \wedge (A_\lambda)_{\mathfrak{p}}] = 0$$

so that, when $\lambda \neq 0$, F_λ frames a new curved flat $\phi_\lambda : M \to G/K$. Moreover, this construction is independent of our original choice of frame F:

Exercise 3.1. If F and $\hat{F} = Fk$ are two frames of a curved flat ϕ then $\hat{F}_\lambda = F_\lambda k$.

In fact, the only ambiguity in our construction comes from the possibility of left multiplying each F_λ by a constant $c_\lambda \in G$. Thus, the curved flats ϕ_λ are defined up to the action of G on G/K.

Note that since $A_1 = A$, we may take $F_1 = F$ and so $\phi_1 = \phi$. Similarly, since A_0 is $\mathfrak{k}$-valued, F_0 may be taken to be K-valued so that ϕ_0 is constant.

To summarise:

Theorem 3.2. *Let* $\phi : M \to G/K$ *be a curved flat with* M *simply connected. Then, for each* $\lambda \in \mathbb{R}$, *there is a map* $\phi_\lambda : M \to G/K$, *uniquely determined up to the action of* G, *such that*

1. *For* $\lambda \in \mathbb{R}^\times$, ϕ_λ *is a curved flat;*
2. $\phi_1 = \phi$;
3. ϕ_0 *is constant.*

We say that the ϕ_λ comprise the *associated family* of ϕ.

So far, our discussion requires no special choice of frame. However, special choices are available and useful: if F frames a curved flat ϕ then (3.4a) says that $A_{\mathfrak{k}}$ solves the Maurer–Cartan equations so that there is a map $k : M \to K$ with $k^{-1}\mathrm{d}k = A_{\mathfrak{k}}$. We now have a new frame $\hat{F} = Fk^{-1}$ of ϕ with

$$\hat{F}^{-1}\mathrm{d}\hat{F} = \mathrm{Ad}\, k(A - k^{-1}\mathrm{d}k) = \mathrm{Ad}(k)A_{\mathfrak{p}} \in \Omega^1 \otimes \mathfrak{p}.$$

This prompts:

Definition. A *flat frame* of a curved flat is a frame F with $F^{-1}\mathrm{d}F \in \Omega^1 \otimes \mathfrak{p}$.

Note that if F is a flat frame then so is each of the F_λ, $\lambda \neq 0$:

$$F_\lambda^{-1}\mathrm{d}F_\lambda = \lambda F^{-1}\mathrm{d}F,$$

while F_0 is constant.

So let F be a flat frame of a curved flat with $F^{-1}\mathrm{d}F = A_{\mathfrak{p}}$. The Maurer–Cartan equations (3.3) read

$$\mathrm{d}A_{\mathfrak{p}} = 0$$
$$[A_{\mathfrak{p}} \wedge A_{\mathfrak{p}}] = 0.$$

We can therefore integrate to get a function $\psi : M \to \mathfrak{p}$ with $\mathrm{d}\psi = A_{\mathfrak{p}}$ and thus

(3.5) $$[\mathrm{d}\psi \wedge \mathrm{d}\psi] = 0.$$

Definition. An immersion $\psi : M \to \mathfrak{p}$ is $\mathfrak{p}$-flat if it satisfies (3.5).

Thus any flat frame gives rise to a $\mathfrak{p}$-flat map and, conversely, a $\mathfrak{p}$-flat map $\psi : M \to \mathfrak{p}$ gives rise to a 1-parameter family of flat frames F_λ framing an associated family of curved flats by solving

$$F_\lambda^{-1}\mathrm{d}F_\lambda = \lambda \mathrm{d}\psi$$

for $\lambda \in \mathbb{R}^\times$.

While we will mostly work with flat frames, we remark that there is another canonical choice of frame for curved flats. For this, we must assume that all $\mathrm{d}\phi(T_pM)$ are conjugate to a fixed semisimple abelian subalgebra $\mathfrak{a} \subset \mathfrak{p}$ (this is certainly the case when each $\mathrm{d}\phi(T_pM)$ is *maximal* abelian and G/K is a Riemannian symmetric space of semisimple type[28]). In this case, one can find a frame for which each $A_{\mathfrak{p}}(T_pM) = \mathfrak{a}$ and then one can prove:

1. $\mathrm{d}A_{\mathfrak{p}} = 0$ so that, for any basis $H_1, \ldots, H_l$ of $\mathfrak{a}$, there are coordinates $x_1, \ldots, x_l$ on M such that $A_{\mathfrak{p}} = \sum_i H_i \mathrm{d}x_i$;
2. There is a unique function $u : M \to [\mathfrak{a}, \mathfrak{k}] \subset \mathfrak{p}$ such that

$$A_{\mathfrak{k}} = [A_{\mathfrak{p}}, u].$$

The Maurer–Cartan equations for A reduce to a differential equation for u called the *l-dimensional system associated to G/K* [65]. This frame is the basis of the approach to curved flats adopted by Terng and her collaborators [6, 65, 66, 67].

---

[28]Thus G is semisimple and K is compact.

3.2. Curved flats and isothermic surfaces.

3.2.1. *The symmetric space $S^n \times S^n \setminus \Delta$.* Denote by Z the space $S^n \times S^n \setminus \Delta$ of pairs of distinct points of $S^n = \mathbb{R}^n \cup \{\infty\}$. There is a diagonal action of $O^+(n+1, 1)$ (and so $SL(\Gamma_n)$) on Z:

$$g(x, y) = (g \cdot x, g \cdot y).$$

Exercise 3.2. Show that this action is transitive.

Let $K \subset SL(\Gamma_n)$ be the stabiliser of $(0, \infty) \in Z$. From (1.5) we see that K is precisely the subgroup of diagonal matrices in $SL(\Gamma_n)$:

$$K = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-t} \end{pmatrix} : a \in \Gamma_n \right\}$$

which is the fixed set of the automorphism τ of $SL(\Gamma_n)$ given by conjugation by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in Pin(n+1, 1)$. The corresponding decomposition $\emptyset = \mathfrak{k} + \mathfrak{p}$ is the familiar decomposition into diagonal and off-diagonal matrices:

$$\mathfrak{k} = \left\{ \begin{pmatrix} \xi & 0 \\ 0 & -\xi^t \end{pmatrix} : \xi \in [\mathbb{R}^n, \mathbb{R}^n] \oplus \mathbb{R} \right\} \qquad \mathfrak{p} = \left\{ \begin{pmatrix} 0 & x \\ y & 0 \end{pmatrix} : x, y \in \mathbb{R}^n \right\}.$$

Finally, $gK \mapsto (g \cdot 0, g \cdot \infty)$ is a diffeomorphism so that Z is identified with the symmetric space $SL(\Gamma_n)/K$.

Remark. There is another model for the symmetric space Z: it can be viewed as the Grassmannian of oriented $(1, 1)$-planes in $\mathbb{R}^{n+1,1}$. Indeed, any pair of distinct points in $\mathbb{P}(L)$ span such a plane while any such plane contains a unique pair of light-lines which are ordered via the orientation.

3.2.2. *Curved flats are Darboux pairs.* A map $\phi : M \to Z = S^n \times S^n \setminus \Delta$ is the same as a pair of maps $f, \hat{f} : M \to S^n$ whose values never coincide. Use the identification of Z with $SL(\Gamma_n)/K$ to view ϕ as a map into $SL(\Gamma_n)/K$ and let $F : M \to SL(\Gamma_n)$ be a frame of ϕ. Then

$$(f, \hat{f}) = (F \cdot 0, F \cdot \infty)$$

so that F frames the pair $(f, \hat{f})$ in the sense of Section 1.3.3. Now let

$$A = F^{-1} dF = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$$

so that

$$A_{\mathfrak{p}} = \begin{pmatrix} 0 & \beta \\ \gamma & 0 \end{pmatrix}$$

for $\beta, \gamma \in \Omega^1 \otimes \mathbb{R}^n$. The curved flat condition $[A_{\mathfrak{p}} \wedge A_{\mathfrak{p}}] = 0$ amounts to $\beta \wedge \gamma = 0$ which, as long as $f, \hat{f}$ are immersions, is precisely the condition of Theorem 2.10 that $(f, \hat{f})$ be a Darboux pair of isothermic surfaces[29].

Say that a map $(f, \hat{f}) : M \to Z$ is *non-degenerate* if both f and $\hat{f}$ are immersions and conclude:

Theorem 3.3. *A non-degenerate map $(f, \hat{f}) : M \to Z$ is a curved flat if and only if $(f, \hat{f})$ is a Darboux pair of isothermic surfaces.*

---

[29]Lemma 1.6 tells us that with $\beta \wedge \gamma = 0$, rank $\beta = 2$. But rank $\beta = $ rank df so dim $M = 2$.

3.2.3. *Spectral deformation is T-transform.* Given a Darboux pair $\phi = (f, \hat{f})$, Theorems 3.3 and 3.2 provide us with the 1-parameter associated family $\phi_\lambda = (f_{(\lambda)}, \hat{f}_{(\lambda)})$ of such with $(f_{(1)}, \hat{f}_{(1)}) = (f, \hat{f})$. In fact, these new isothermic surfaces are T-transforms of f and $\hat{f}$. To see this, fix a polarisation Q and thus a Christoffel transform f^c of f so that $\hat{f} = \mathcal{D}_r f$ for some $r \in \mathbb{R}^\times$. As usual, take

$$F = \begin{pmatrix} \hat{f}g^{-1} & f \\ g^{-1} & 1 \end{pmatrix}$$

so that

$$A_{\mathfrak{p}} = \begin{pmatrix} 0 & \mathrm{d}f \\ -r\mathrm{d}f^c & 0 \end{pmatrix}.$$

Then $(f_{(\lambda)}, \hat{f}_{(\lambda)})$ is framed by $F_\lambda : M \to \mathrm{SL}(\Gamma_n)$ with

$$F_\lambda^{-1}\mathrm{d}F_\lambda = A_{\mathfrak{k}} + \lambda A_{\mathfrak{p}}.$$

Now replace F_λ with the frame $F_\lambda \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & 1/\sqrt{\lambda} \end{pmatrix}$ which has Maurer–Cartan form

$$A_{\mathfrak{k}} + \begin{pmatrix} 0 & \mathrm{d}f \\ -\lambda^2 r\mathrm{d}f^c & 0 \end{pmatrix} = A_{\mathfrak{k}} + A_{\mathfrak{p}} + (1 - \lambda^2)r \begin{pmatrix} 0 & 0 \\ \mathrm{d}f^c & 0 \end{pmatrix}.$$

Thus, by Theorem 2.12, $f_{(\lambda)} = \mathcal{T}_{(1-\lambda^2)r} f$.

Exercise 3.3. Contemplate the frame

$$F_\lambda \begin{pmatrix} 0 & -1/\sqrt{\lambda} \\ \sqrt{\lambda} & 0 \end{pmatrix}$$

of $(\hat{f}_{(\lambda)}, f_{(\lambda)})$ to conclude that $\hat{f}_{(\lambda)} = \mathcal{T}_{(1-\lambda^2)r} \hat{f}$.

To summarise:

Theorem 3.4. *The associated family of a Darboux pair $(f, \hat{f})$ consists of T-transforms of the pair:*

$$f_{(\lambda)} = \mathcal{T}_{(1-\lambda^2)r} f, \qquad \hat{f}_{(\lambda)} = \mathcal{T}_{(1-\lambda^2)r} \hat{f}$$

Remark. The extraction of roots in our gauge transformations means we must take $\lambda > 0$. However, since

$$\tau A_\lambda = A_{\mathfrak{k}} - \lambda A_{\mathfrak{p}} = A_{-\lambda},$$

τF_λ and $F_{-\lambda}$ differ by a constant so that the pairs $(f_{(\lambda)}, \hat{f}_{(\lambda)})$ and $(f_{(-\lambda)}, \hat{f}_{(-\lambda)})$ differ by a Möbius transformation. We shall have more to say about this symmetry below.

3.2.4. $\mathfrak{p}$-*flat maps are Christoffel pairs.* The alert reader will have noticed by now that there is a second way to construct a pair of isothermic surfaces from a curved flat: the Maurer–Cartan form of a flat frame of a curved flat is the derivative of a $\mathfrak{p}$-flat map $\psi : M \to \mathfrak{p}$:

$$A_{\mathfrak{p}} = \mathrm{d}\psi.$$

In our case, write

$$\psi = \begin{pmatrix} 0 & f_0 \\ f_0^c & 0 \end{pmatrix}$$

for $f_0, f_0^c : M \to \mathbb{R}^n$. Then $[\mathrm{d}\psi \wedge \mathrm{d}\psi] = 0$ amounts to

$$\mathrm{d}f_0 \wedge \mathrm{d}f_0^c = 0$$

and its transpose so that a $\mathfrak{p}$-map is precisely a dual pair of isothermic surfaces!

It is important to emphasise that this pair is *not* the Darboux pair comprising the curved flat. Rather, the two pairs are T-transforms of each other: indeed, if the flat frame F frames the Darboux pair $(f, \hat{f})$ we have

$$F^{-1} dF = \begin{pmatrix} 0 & df_0 \\ df_0^c & 0 \end{pmatrix}$$

so that $f = \mathcal{T}_1 f_0$ and, by Theorem 2.14, $\hat{f} = \mathcal{T}_1 f_0^c$.

Conversely, given a Christoffel pair (f_0, f_0^c), we integrate to obtain the associated family of flat frames F_λ with

$$F_\lambda^{-1} dF_\lambda = \lambda \begin{pmatrix} 0 & df_0 \\ df_0^c & 0 \end{pmatrix}.$$

The F_λ frame Darboux pairs $(f_{(\lambda)}, \hat{f}_{(\lambda)})$ and we argue as in Section 3.2.3 to prove:

Exercise 3.4. $f_{(\lambda)} = \mathcal{T}_{\lambda^2} f_0$, $\hat{f}_{(\lambda)} = \mathcal{T}_{\lambda^2} f_0^c$.

As we shall see in Section 4.1, if the constants of integration are chosen correctly, we can recover (f_0, f_0^c) up to a translation from the frames F_λ via the Sym formula [24]:

$$\begin{pmatrix} 0 & f_0 \\ f_0^c & 0 \end{pmatrix} = \frac{\partial F_\lambda}{\partial \lambda}\Bigg|_{\lambda=0}$$

so that a Christoffel transform is a limit of Darboux transforms.

In conclusion, we have seen that an associated family of curved flats in Z amounts to the family of T-transforms (for $r > 0$) of a Christoffel pair of isothermic surfaces, each T-transform being, as we know from Theorem 2.11, a Darboux pair of isothermic surfaces. However, the curved flat formulation gives us more: curved flats admit a zero curvature formulation which means that we can apply the powerful loop group approach to integrable systems and, in doing so, find a completely different view-point on the topics we have been studying. It is to this that we now turn.

4. Loop groups and Bäcklund transformations

We are going to show that associated families of curved flats (or rather their flat frames) are the same as certain maps into an infinite dimensional group $\mathcal{G}^+$ of holomorphic maps from $\mathbb{C}$ into a complex Lie group. Completely general principles, first enunciated by Zakharov and his collaborators [74, 75], then allow us to construct a local action of a second infinite-dimensional group $\mathcal{G}^-$ on these families. In general, computation of this action amounts to solving a Riemann–Hilbert problem but, as has been made clear in a series of papers by Terng and Uhlenbeck [66, 67, 70, 71], the action of certain elements of $\mathcal{G}^-$, the *simple factors*, can be computed explicitly. In several geometric problems, the action of these simple factors amount to known Bäcklund transformations.

We shall show that this is the case for isothermic surfaces: the action of simple factors will turn out to be precisely by Darboux transforms of the underlying Christoffel pair. This places our theory in a well-understood context in integrable systems theory and, in particular, general arguments of Terng–Uhlenbeck [67] can be exploited to establish Bianchi permutability of Darboux transforms. In this way, we find a second approach to the results of Section 2.4.

4.1. **Extended flat frames.** Henceforth M will be simply connected with a fixed base-point $o \in M$.

Let G/K be a symmetric space. Further let $G^{\mathbb{C}}$ be the complexification of G and denote by $g \mapsto \bar{g}$ the conjugation across the real form G. Thus $g \mapsto \bar{g}$ is the anti-holomorphic involution on $G^{\mathbb{C}}$ with fixed set G.

Let $\psi : M \to \mathfrak{p}$ be a $\mathfrak{p}$-flat map and set $A_{\mathfrak{p}} = \mathrm{d}\psi$. We have already seen how ψ gives rise to a family of flat frames F_{λ} with

$$F_{\lambda}^{-1}\mathrm{d}F_{\lambda} = \lambda A_{\mathfrak{p}},$$

for $\lambda \in \mathbb{R}$. We now extend this construction to $\lambda \in \mathbb{C}$ and fix the constants of integration: for $\lambda \in \mathbb{C}$, let $F_{\lambda} : M \to G^{\mathbb{C}}$ be the *unique* map with

$$F_{\lambda}^{-1}\mathrm{d}F_{\lambda} = \lambda A_{\mathfrak{p}}$$
$$F_{\lambda}(o) = 1.$$

The existence of each F_{λ} is guaranteed since $\lambda A_{\mathfrak{p}}$ solves the Maurer–Cartan equations and M is simply connected.

We note:

1. $F_0 = 1$ since $F_0^{-1}\mathrm{d}F_0 = 0$ and $F_0(o) = 1$.
2. For each $p \in M$, $\lambda \mapsto F_{\lambda}(p) : \mathbb{C} \to G^{\mathbb{C}}$ is holomorphic since $\lambda \mapsto \lambda A_{\mathfrak{p}}$ is certainly holomorphic as is $\lambda \mapsto F_{\lambda}(o)$.
3. For all $\lambda \in \mathbb{C}$,
$$\overline{F_{\lambda}} = F_{\bar{\lambda}}$$
   or, equivalently, $F_{\lambda} : M \to G$ when $\lambda \in \mathbb{R}$. This holds since
$$\overline{\lambda A_{\mathfrak{p}}} = \bar{\lambda} A_p$$
   so that $\overline{F_{\lambda}}$ and $F_{\bar{\lambda}}$ have the same Maurer–Cartan form and the same value at o and so must coincide.
4. Similarly, since $\tau(\lambda A_{\mathfrak{p}}) = -\lambda A_{\mathfrak{p}}$, we conclude that, for all $\lambda \in \mathbb{C}$,
$$\tau F_{\lambda} = F_{-\lambda}.$$

We now change our point of view and assemble the F_{λ} into a single map $\Phi : M \to \mathrm{Map}(\mathbb{C}, G^{\mathbb{C}})$ by setting

$$\Phi(p)(\lambda) = F_{\lambda}(p).$$

Observe that Φ takes values in the group $\mathcal{G}^+$ of *holomorphic* maps $g : \mathbb{C} \to G^{\mathbb{C}}$ satisfying

(4.1a) $g(0) = 1,$

(4.1b) $\tau g(\lambda) = g(-\lambda),$

(4.1c) $\overline{g(\lambda)} = g(\bar{\lambda}),$

for all $\lambda \in \mathbb{C}$. It is easy to see that $\mathcal{G}^+$ is a group under point-wise multiplication.

Definition. A map $\Phi : M \to \mathcal{G}^+$ is an *extended flat frame* if and only if

(4.2) $\Phi^{-1}\mathrm{d}\Phi(\lambda) = \lambda A_{\mathfrak{p}}$

with $A_{\mathfrak{p}} \in \Omega^1 \otimes \mathfrak{p}$ independent of λ.

Φ is additionally said to be *based* if $\Phi(o) = 1$.

The property of being an extended flat frame is characterised entirely by the behaviour at $\lambda = \infty$ of $\Phi^{-1}\mathrm{d}\Phi$:

Lemma 4.1. $\Phi : M \to \mathcal{G}^+$ *is an extended flat frame if and only if, for each* $p \in M$, $\Phi^{-1}\mathrm{d}\Phi_{|p}$ *has a simple pole at* $\lambda = \infty$.

Proof. Let $\Phi : M \to \mathcal{G}^+$ and contemplate the power series expansion of $\Phi^{-1}\mathrm{d}\Phi$:

$$\Phi^{-1}\mathrm{d}\Phi = \sum_{n \geq 0} \lambda^n A_n$$

with $A_n \in \Omega^1 \otimes \mathfrak{g}^{\mathbb{C}}$. The twisting and reality conditions (4.1b) and (4.1c) force

$$\tau \sum_{n \geq 0} \lambda^n A_n = \sum_{n \geq 0} (-\lambda)^n A_n, \qquad \overline{\sum_{n \geq 0} \lambda^n A_n} = \sum_{n \geq 0} \bar{\lambda}^n A_n$$

whence

$$A_{2n} \in \Omega^1 \otimes \mathfrak{k}, \qquad A_{2n-1} \in \Omega^1 \otimes \mathfrak{p}.$$

Moreover, $\Phi(0) \equiv 1$ so that $A_0 = 0$.

Thus $\Phi^{-1}\mathrm{d}\Phi$ has a simple pole at $\lambda = \infty$ if and only if all $A_n = 0$ for $n > 1$ which is the case precisely when $\Phi^{-1}\mathrm{d}\Phi = \lambda A_1$ for some $A_1 \in \Omega^1 \otimes \mathfrak{p}$. $\qquad\square$

We can recover the generating $\mathfrak{p}$-flat map up to translation from Φ by a popular device known as the Sym formula:

Proposition 4.2. *Let* Φ *be an extended flat frame with* $\Phi^{-1}\mathrm{d}\Phi = \lambda A_{\mathfrak{p}}$ *and define* $\psi_0 : M \to \mathfrak{g}$ *by*

$$(4.3) \qquad\qquad\qquad \psi_0 = \left.\frac{\partial \Phi}{\partial \lambda}\right|_{\lambda=0}$$

Then

1. $\psi_0 : M \to \mathfrak{p}$;
2. $\mathrm{d}\psi_0 = A_{\mathfrak{p}}$.

Proof. We have $\tau\Phi(\lambda) = \Phi(-\lambda)$ and differentiating with respect to λ gives

$$\tau\left.\frac{\partial \Phi}{\partial \lambda}\right|_{\lambda=0} = -\left.\frac{\partial \Phi}{\partial \lambda}\right|_{\lambda=0},$$

that is, $\psi_0 : M \to \mathfrak{p}$.

Now view Φ as a map $M \times \mathbb{C} \to G^{\mathbb{C}}$ with Maurer–Cartan form α. Then, for $p \in M$ and $X \in T_pM$, we have

$$\alpha_{(p,0)}(\partial/\partial\lambda) = \psi_0(p);$$
$$\alpha_{(p,\lambda)}(X) = \lambda A_{\mathfrak{p}}(X).$$

The Maurer–Cartan equations for α give

$$\mathrm{d}\alpha_{(p,0)}(\partial/\partial\lambda, X) + [\alpha_{(p,0)}(\partial/\partial\lambda), \alpha_{(p,0)}(X)] = 0.$$

However, $\alpha_{(p,0)}(X) = 0$ since $\Phi(p)(0) = 1$ for all $p \in M$ so we are left with

$$\left.\frac{\partial\alpha(X)}{\partial\lambda}\right|_{\lambda=0} - \mathrm{d}_X\alpha(\partial/\partial\lambda) = 0,$$

that is, $A_{\mathfrak{p}}(X) = \mathrm{d}_X\psi_0$. $\qquad\square$

Thus, if $\psi : M \to \mathfrak{p}$ is a $\mathfrak{p}$-flat map and Φ is the corresponding *based* extended flat frame, then

$$\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} \Phi(o) = 0$$

so that

$$(4.4) \qquad \psi = \left.\frac{\partial\Phi}{\partial\lambda}\right|_{\lambda=0} + \psi(o).$$

This gives us a bijective correspondence:

$$\{\mathfrak{p}\text{-flat maps}\} \to \{\text{based extended flat frames}\} \times \mathfrak{p}$$
$$\psi \mapsto (\Phi, \psi(o))$$

with inverse

$$(\Phi, \xi) \mapsto \left.\frac{\partial\Phi}{\partial\lambda}\right|_{\lambda=0} + \xi.$$

The Sym formula has geometric content: for $\lambda \in \mathbb{R}$, let $\phi_\lambda : M \to G/K$ be the curved flat framed by $\Phi(\lambda)$. In particular $\phi_0 \equiv eK$, the identity coset. With the usual identification $T_{eK}G/K \cong \mathfrak{p}$, one sees that

$$\psi_0 = \left.\frac{\partial\phi_\lambda}{\partial\lambda}\right|_{\lambda=0}.$$

In particular, in the isothermic surface case, we have $\mathfrak{p} \cong T_0S^n \oplus T_\infty S^n$ and an associated family of Darboux pairs $(f_{(\lambda)}, \hat{f}_{(\lambda)})$ with

$$f_{(0)} \equiv 0, \qquad \hat{f}_{(0)} \equiv \infty.$$

The generating Christoffel pair (f, f^c) are recovered by "blowing up" their T-transforms as $\lambda \to 0$:

$$f = \left.\frac{\partial f_{(\lambda)}}{\partial\lambda}\right|_{\lambda=0} : M \to T_0S^n;$$

$$\hat{f} = \left.\frac{\partial\hat{f}_{(\lambda)}}{\partial\lambda}\right|_{\lambda=0} : M \to T_\infty S^n.$$

4.2. **The dressing action.** We are going to define a local action of a group of rational maps on the set of extended flat frames and so, eventually, on the set of $\mathfrak{p}$-flat maps. Our action will be by point-wise application of a local action on $\mathcal{G}^+$ which we now describe.

Let $\mathcal{G}$ denote the group of holomorphic maps $g : \mathrm{dom}(g) \subset \mathbb{P}^1 \to G^{\mathbb{C}}$ of affine subsets of the Riemann sphere which are twisted and real in the sense that

$$(4.5\text{a}) \qquad\qquad \tau g(\lambda) = g(-\lambda),$$

$$(4.5\text{b}) \qquad\qquad \overline{g(\lambda)} = g(\bar{\lambda}),$$

for all $\lambda \in \mathrm{dom}(g)$. Clearly $\mathcal{G}^+$ is a subgroup of $\mathcal{G}$. We define a second subgroup $\mathcal{G}^-$ by

$$\mathcal{G}^- = \{g \in \mathcal{G} : g \text{ is rational on } \mathbb{P}^1 \text{ and holomorphic near } \infty\}.$$

Thus $\mathcal{G}^+$ consists of those elements of $\mathcal{G}$ which are holomorphic on $\mathbb{C}$ while $\mathcal{G}^-$ consists of those which are rational[30] and holomorphic near ∞.

---

[30] The restriction to rational maps is not really necessary: one could work with the group of germs at ∞ of maps to $G^{\mathbb{C}}$ with (4.5). While not appropriate here, such generality is necessary in some contexts, see [12] for a discussion in a related situation.

Lemma 4.3. $\mathcal{G}^+ \cap \mathcal{G}^- = \{1\}$.

Proof. If $g \in \mathcal{G}^+ \cap \mathcal{G}^-$ then g is holomorphic on $\mathbb{P}^1$ and so is constant. Moreover $g(0) = 1$ whence $g = 1$. $\qquad\square$

The basis of our action is the Birkhoff-Grothendieck decomposition theorem in a formulation due to Pressley–Segal [59]:

Theorem 4.4. *Set* $\mathcal{U} = \mathcal{G}^+\mathcal{G}^-$. *Then* $\mathcal{U}$ *is a dense open*[31] *subset of* $\mathcal{G}$.

Thus $g \in \mathcal{U}$ if and only if we can write

$$(4.6) \qquad\qquad g = g_+ g_-$$

with $g_\pm \in \mathcal{G}^\pm$.

Exercise 4.1. Use Lemma 4.3 to show that the decomposition (4.6) is unique when it exists.

For $g_- \in \mathcal{G}^-$, set $\mathcal{U}_{g_-} = g_-^{-1}\mathcal{U}g_- \cap \mathcal{G}^+$: this is an open neighbourhood of 1 in $\mathcal{G}^+$.

Lemma 4.5. $g_+ \in \mathcal{U}_{g_-}$ *if and only if there are unique* $\hat{g}_\pm \in \mathcal{G}^\pm$ *such that*

$$(4.7) \qquad\qquad g_- g_+ = \hat{g}_+ \hat{g}_-$$

on $\mathbb{C} \cap \mathrm{dom}(g_-)$.

Proof. If (4.7) holds then

$$g_+ = g_-^{-1}\hat{g}_+\hat{g}_- = g_-^{-1}\hat{g}_+\hat{g}_- g_-^{-1}g_- \in g_-^{-1}\mathcal{G}^+\mathcal{G}^- g_- \cap \mathcal{G}^+ = \mathcal{U}_{g_-}.$$

Conversely, if $g_+ \in \mathcal{U}_{g_-}$ then $g_- g_+ g_-^{-1} \in \mathcal{U}$ so we can write

$$g_- g_+ g_-^{-1} = h_+ h_-$$

with $h_\pm \in \mathcal{G}^\pm$. Now put $\hat{g}_+ = h_+$ and $\hat{g}_- = h_- g_-$.

The uniqueness assertion is proved as in Exercise 4.1. $\qquad\square$

Notation. Write $g_-\#g_+$ for $\hat{g}_+$ in (4.7).

Thus $g_-\#g_+ = g_- g_+ \hat{g}_-^{-1}$.

Exercise 4.2. Show:

1. $\mathcal{U}_1 = \mathcal{G}^-$ and $1\#g_+ = g_+$ for all $g_+ \in \mathcal{G}^+$.
2. For all $g_- \in \mathcal{G}^-$, $g_-\#1 = 1$.

Now let $g_1, g_2 \in \mathcal{G}^-$, $g_+ \in \mathcal{U}_{g_1}$ and suppose $g_1\#g_+ \in \mathcal{U}_{g_2}$ so that $g_2\#(g_1\#g_+)$ is defined. This means we have $\hat{g}_1, \hat{g}_2 \in \mathcal{G}^-$ such that

$$g_2(g_1 g_+ \hat{g}_1^{-1})\hat{g}_2^{-1} = g_2\#(g_1\#g_+) \in \mathcal{G}^+$$

whence

$$(g_2 g_1)g_+ = (g_2\#(g_1\#g_+))\hat{g}_2\hat{g}_1.$$

Since $\hat{g}_2\hat{g}_1 \in \mathcal{G}^-$, we conclude that $g_+ \in \mathcal{U}_{g_2 g_1}$ and that $(g_2 g_1)\#g_+ = g_2\#(g_1\#g_+)$. Taking this together with Exercise 4.2, we conclude:

Theorem 4.6. $g_-\#g_+$ *defines a local action of* $\mathcal{G}^-$ *on* $\mathcal{G}^+$.

---

[31]The reader may object that I have not topologised $\mathcal{G}$: in fact, the compact open topology will do (or any stronger one).

Now let $\Phi : M \to \mathcal{G}^+$ be a map and $g_- \in \mathcal{G}^-$. Define $g_-\#\Phi : \Phi^{-1}(\mathcal{U}_{g_-}) \subset M \to \mathcal{G}^+$ by

$$(g_-\#\Phi)(p) = g_-\#\big(\Phi(p)\big).$$

The whole point of this is contained in the following theorem:

Theorem 4.7. *If $\Phi : M \to \mathcal{G}^+$ is a (based) extended flat frame then so is $g_-\#\Phi$.*

Proof. Set $\hat{\Phi} = g_-\#\Phi$ and let $\hat{A}$ be its Maurer–Cartan form. By Lemma 4.1, we must show that $\hat{A}$ has a simple pole at $\lambda = \infty$. However, in a punctured neighbourhood of ∞, we have

$$\hat{\Phi} = g_-\Phi\hat{g}_-^{-1}$$

with $\hat{g}_-^{-1} : \Phi^{-1}(\mathcal{U}_{g_-}) \to \mathcal{G}^-$ so that

$$\hat{A} = \operatorname{Ad}\hat{g}_-(\Phi^{-1}\mathrm{d}\Phi) - \mathrm{d}\hat{g}_-\hat{g}_-^{-1}.$$

Since $\hat{g}_-$ is holomorphic at $\lambda = \infty$, we immediately conclude that $\hat{A}$ has the same pole at ∞ as $\Phi^{-1}\mathrm{d}\Phi$.

Finally, $g_-\#1 = 1$ so that $\hat{\Phi}$ is based if and only if Φ is. $\qquad\square$

Remark. To get this far, we have used very little of the specifics of the situation. To get a local action of $\mathcal{G}^-$ on $\mathcal{G}^+$ we only used that $\mathcal{G}^+\mathcal{G}^-$ is open in $\mathcal{G}$ with $\mathcal{G}^+ \cap \mathcal{G}^- = \{1\}$. Moreover the argument of Theorem 4.7 is also very general: the only ingredient is that membership of the class of extended frames is determined by the pole behaviour of the Maurer–Cartan form. For then, the pointwise action of the group of maps holomorphic near these poles will preserve that class. Thus one can use exactly the same techniques to produce actions of such groups in a variety of geometric problems. See [10, 12, 70] among others for the case of harmonic maps and the work of Terng–Uhlenbeck [66, 67] for many other examples.

We would like our action of $\mathcal{G}^-$ on based extended flat frames to induce an action on $\mathfrak{p}$-flat maps. However, since the frame only determines the $\mathfrak{p}$-flat map up to translation, we must work with a slightly smaller group which has an action on $\mathfrak{p}$ also.

For this, define $\mathcal{G}_*^- \subset \mathcal{G}^-$ by

$$\mathcal{G}_*^- = \{g \in \mathcal{G}^- : g(0) = 1\}.$$

For $g_- \in \mathcal{G}_*^-$ and $\psi : M \to \mathfrak{p}$ a $\mathfrak{p}$-flat map, let Φ be the based extended flat frame with $\Phi^{-1}\mathrm{d}\Phi = \lambda\mathrm{d}\psi$ and define $g_-\#\psi : \Phi^{-1}(\mathcal{U}_{g_-}) \to \mathfrak{p}$ by

$$(4.8) \qquad g_-\#\psi = \psi(o) + \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0}(g_-\#\Phi) - \left.\frac{\partial g_-}{\partial\lambda}\right|_{\lambda=0}.$$

Note that $\partial g_-/\partial\lambda|_{\lambda=0} \in \mathfrak{p}$ so that the right hand side is indeed $\mathfrak{p}$-valued.

$g_-\#\psi$ differs from $\partial/\partial\lambda|_{\lambda=0}(g_-\#\Phi)$ by constants so that, by Proposition 4.2, $g_-\#\psi$ is again a $\mathfrak{p}$-flat map.

Exercise 4.3. Show that, for ψ a $\mathfrak{p}$-flat map and $g_1, g_2 \in \mathcal{G}_*^-$,

$$1\#\psi = \psi$$
$$g_1\#(g_2\#\psi) = (g_1g_2)\#\psi$$

whenever the left hand side is defined.

Thus we conclude:

Theorem 4.8. *There is a local action of $\mathcal{G}_*^-$ on $\mathfrak{p}$-flat maps given by (4.8).*

We obtain a more efficient formula for this action as follows: write

$$g_-\Phi = (g_-\#\Phi)\hat{g}_-$$

with $\hat{g}_- : \Phi^{-1}(\mathcal{U}_{g_-}) \to \mathcal{G}_*^-$ so that

$$g_-\#\Phi = g_-\Phi\hat{g}_-^{-1}.$$

Thus

$$\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0}(g_-\#\Phi) = \left.\frac{\partial g_-}{\partial\lambda}\right|_{\lambda=0} + \left.\frac{\partial\Phi}{\partial\lambda}\right|_{\lambda=0} + \left.\frac{\partial\hat{g}_-^{-1}}{\partial\lambda}\right|_{\lambda=0}.$$

Now (4.4) gives

$$\left.\frac{\partial\Phi}{\partial\lambda}\right|_{\lambda=0} = \psi - \psi(o)$$

and feeding all this into (4.8) gives:

(4.9)
$$g_-\#\psi = \psi + \left.\frac{\partial\hat{g}_-^{-1}}{\partial\lambda}\right|_{\lambda=0}.$$

4.3. **Simple factors.** Given $g_- \in \mathcal{G}^-$ and Φ an extended flat frame, a basic problem is to compute $g_-\#\Phi$. This amounts to performing the factorisation

$$g_-\Phi(p) = \hat{g}_+(p)\hat{g}_-(p)$$

for each $p \in M$ and, in general, this is a Riemann–Hilbert problem. Part of the philosophy of Terng–Uhlenbeck is that there are special elements of $\mathcal{G}^-$, the *simple factors*, for which one can explicitly perform the factorisation by algebra alone and, moreover, that the action of these factors amount to Bäcklund-type transformations of the underlying geometric problem. Of course, the Art in this approach is to put one's hands on these simple factors!

Let us look for some hints. We are given $g_\pm \in \mathcal{G}^\pm$ and seek $\hat{g}_\pm \in \mathcal{G}^\pm$ so that

(4.10)
$$g_-g_+ = \hat{g}_+\hat{g}_-.$$

First observe that any $g_- \neq 1 \in \mathcal{G}_*^-$ must have some singularities in $\mathbb{C}^\times$, that is, $\lambda \in \mathbb{C}^\times$ where g either fails to be defined or fails to be invertible. In view of the twisting and reality conditions (4.5), if α is such a singularity, so is $-\alpha$ and $\bar{\alpha}$.

Secondly, rearrange (4.10) to get

$$\hat{g}_- = \hat{g}_+^{-1}g_-g_+$$

with both $g_+, \hat{g}_+$ holomorphic on $\mathbb{C}$. Thus $\hat{g}_-$ has the same singularities as g_-.

The idea now is to work with g_- having the minimum number of singularities. In our case, this number is two and we are contemplating g_- with poles at $\pm\alpha$ and demand that either $\bar{\alpha} = \alpha$ so that $\alpha \in \mathbb{R}$ or $\bar{\alpha} = -\alpha$ so that $\alpha \in \sqrt{-1}\mathbb{R}$.

To get further, we begin by considering the case where G is compact. Here the situation reduces to one which is completely understood. When G is compact, we can have no singularities on $\mathbb{R}$ and thus $\alpha \in \sqrt{-1}\mathbb{R}$. Now use a linear fractional transformation to move the singularities at $\pm\alpha$ to 0 and ∞: define $t_\alpha : \mathbb{P}^1 \to \mathbb{P}^1$ by

$$t_\alpha(\lambda) = \frac{\alpha - \lambda}{\alpha + \lambda}$$

so that

$$t_\alpha(\alpha) = 0, \quad t_\alpha(-\alpha) = \infty, \quad t_\alpha(0) = 1$$
$$t_\alpha(\mathbb{R}) \subset S^1 = \{\lambda : |\lambda| = 1\}$$

We write
$$g_- = h_- \circ t_\alpha$$
for $h_- : \mathbb{C}^\times \to G^{\mathbb{C}}$. Since g_- is rational, h_- is a Laurent polynomial and, moreover, we have $h_-(1) = 1$ and $h(S^1) \subset G$. Otherwise said, h_- lies in the based algebraic loop group $\Omega_{\mathrm{alg}}G$ of Laurent polynomial maps $h : \mathbb{C}^\times \to G^{\mathbb{C}}$ satisfying
$$h(1) = 1$$
$$\overline{h(\lambda)} = h(1/\bar{\lambda}).$$
This group features in a factorisation problem which can always be solved: let $\Lambda^+ G^{\mathbb{C}}$ denote the group of maps h_+ to $\mathcal{G}^{\mathbb{C}}$ which are defined and holomorphic near 0 and ∞ and have the reality condition
$$\overline{h_+(\lambda)} = h_+(1/\bar{\lambda}),$$
for $\lambda \in \mathrm{dom}(h_+)$. It follows from the results of Pressley-Segal [59] that, for $h_+ \in \Lambda^+ G^{\mathbb{C}}$, $h_- \in \Omega_{\mathrm{alg}}G$, there is always a unique decomposition
$$h_+ h_- = \hat{h}_- \hat{h}_+$$
with $\hat{h}_- \in \Omega_{\mathrm{alg}}G$ and $\hat{h}_+ \in \Lambda^+ G^{\mathbb{C}}$. Just as before, we set
$$h_+ \cdot h_- = \hat{h}_-$$
to get a (now global) action of $\Lambda^+ G^{\mathbb{C}}$ on $\Omega_{\mathrm{alg}}G$ which is well understood.

The relevance of all this to our own factorisation problem is that, after taking inverses and moving the poles with t_α, our decomposition problem becomes that of Pressley–Segal:

Exercise 4.4. For $g_\pm \in \mathcal{G}^\pm$ with g_- having only singularities at $\pm\alpha$, write
$$g_\pm = h_\pm \circ t_\alpha$$
so that $h_- \in \Omega_{\mathrm{alg}}G$ and $h_+ \in \Lambda^+ G^{\mathbb{C}}$ (since g_+ is holomorphic at $\pm\alpha$). Then
$$g_- \# g_+ = (h_+^{-1} \cdot h_-^{-1})^{-1} \circ t_\alpha^{-1}.$$

In particular, we deduce

Proposition 4.9. *If G is compact and g_- has only two poles then $\mathcal{U}_{g_-} = \mathcal{G}^+$.*

There is more: in this setting, the action of such a g_- is, in principle, computable algebraically:

Fact. *The orbits of $\Lambda^+ G^{\mathbb{C}}$ on $\Omega_{\mathrm{alg}}G$ are finite-dimensional: they form the* Bruhat *decomposition of $\Omega_{\mathrm{alg}}G$ [59]. In fact, $h_+ \cdot h_-$ depends only on a finite jet of h_+ at $\lambda = 0$.*

As a consequence, $g_- \# g_+$ can be computed from g_- and a finite jet at α of g_+. The maximally desirable situation is when only the 0-jet $g_+(\alpha)$ is involved: again, this amounts to a feature of the Bruhat decomposition.

Fact. *[59, 10] $h_+ \cdot h_-$ depends only on $h_+(0)$ if and only if $h_- : \mathbb{C}^\times \to G^C$ is a homomorphism such that $\mathrm{Ad}\, h_- : \mathbb{C}^\times \to \mathrm{Ad}(G^{\mathbb{C}})$ has simple poles only. In this case, $\hat{h}_-$ is another homomorphism in the same (real) conjugacy class.*

We therefore conclude that if we wish to be able to compute $g_- \# \Phi(p)$ from just g_- and the value $\Phi(p)(\alpha)$ then we are compelled to take

$$(4.11) \qquad\qquad g_-(\lambda) = \gamma\left(\frac{\alpha - \lambda}{\alpha + \lambda}\right)$$

where $\operatorname{Ad}\gamma : \mathbb{C}^{\times} \to \operatorname{Ad}(G^{\mathbb{C}})$ is a homomorphism with only simple poles. This last is a very restrictive condition: it implies that the real conjugacy class of γ is a Hermitian symmetric space and so excludes the exceptional Lie groups G_2, F_4 and E_8. in fact, there is more: we need $\tau g_-(\lambda) = g_-(-\lambda)$ which amounts to demanding

$$\tau\gamma(\mu) = \gamma(1/\mu),$$

for $\mu \in \mathbb{C}^{\times}$ and this excludes many symmetric spaces[32].

Be that as it may, for G compact, we have shown that any g_- with the properties we want must be of the form (4.11). For G non-compact, we make this our ansatz:

Definition. $g_- \in \mathcal{G}_*^-$ is a *simple factor* if it is the form

$$g_- = \gamma \circ t_\alpha$$

with $\alpha^2 \in \mathbb{R}$ and $\gamma : \mathbb{C}^{\times} \to G^{\mathbb{C}}$ a homomorphism for which $\operatorname{Ad}\gamma$ has simple poles.

It turns out that simple factors retain their desirable property of having algebraically computable action even for non-compact G. However, to develop the theory any further in this general setting will take us too far afield so we now turn to the case of relevance to isothermic surfaces.

4.4. **Simple factors for $S^n \times S^n \setminus \Delta$.** We are going to classify the simple factors for $G = \mathrm{O}^+(n+1, 1)$ and so begin by determining the homomorphisms $\gamma : \mathbb{C}^{\times} \to \mathrm{O}^+(n+1, 1)^{\mathbb{C}} = \emptyset[n+2, \mathbb{C}]$ for which $\operatorname{Ad}\gamma$ has simple poles.

Let $\gamma : \mathbb{C}^{\times} \to \emptyset[n+2, \mathbb{C}]$ be a homomorphism. There is a decomposition of $\mathbb{C}^{n+2}$ into common eigenspaces of the $\gamma(\lambda)$:

$$\mathbb{C}^{n+2} = \oplus_{i=-k}^{k} V_i$$

so that, with π_i the projection onto V_i along $\oplus_{i \neq j} V_j$, we have

$$\gamma(\lambda) = \sum_{i=-k}^{k} \lambda^i \pi_i$$

(we must allow the possibility that some $V_i = \{0\}$). Since $\gamma(\lambda) \in \emptyset[n+2, \mathbb{C}]$, we have $V_i \perp V_j$ for $i + j \neq 0$ so that each V_i is isotropic for $i \neq 0$, $\dim V_i = \dim V_{-i}$ and $V_0^{\perp} = \oplus_{i \neq 0} V_i$. As $\emptyset[n+2, \mathbb{C}]$-modules, $\emptyset[n+2, \mathbb{C}] \cong \bigwedge^2 \mathbb{C}^{n+2}$ via

$$(u \wedge v)w = (u, w)v - (v, w)u$$

and using this identification we immediately see that $\operatorname{Ad}\gamma(\lambda)$ has eigenvalues λ^{2i} on $\bigwedge^2 V_i$ and λ^{i+j} on $V_i \otimes V_j$, $i \neq j$. Thus $\operatorname{Ad}\gamma$ has simple poles exactly when $k = 1$ and $\dim V_1 = 1$ (to ensure $\bigwedge^2 V_1 = \{0\}$). We are therefore working with γ of the form

$$\gamma(\lambda) = \lambda\pi_+ + \pi_0 + \lambda^{-1}\pi_-$$

corresponding to a decomposition

$$\mathbb{C}^{n+2} = L_+ \oplus L_0 \oplus L_-$$

with $L_\pm$ 1-dimensional isotropic subspaces and $L_0 = (L_+ \oplus L_-)^{\perp}$.

The key to computing the dressing action of the corresponding simple factor is the following lemma:

---

[32]For example, if G/K is a projective space $\mathbb{RP}^n$, $\mathbb{CP}^n$ or $\mathbb{HP}^n$, then such γ exist only when $n = 1$.

Lemma 4.10. *Let* $\gamma(\lambda) = \lambda\pi_+ + \pi_0 + \lambda^{-1}\pi_-$ *and* $\hat{\gamma} = \lambda\hat{\pi}_+ + \hat{\pi}_0 + \lambda^{-1}\hat{\pi}_-$ *be homomorphisms as above with* $\mathrm{Ad}\,\gamma$, $\mathrm{Ad}\,\hat{\gamma}$ *having simple poles and let*

$$\mathbb{C}^{n+2} = L_+ \oplus L_0 \oplus L_- = \hat{L}_+ \oplus \hat{L}_0 \oplus \hat{L}_-$$

be the corresponding eigenspace decompositions.

Let E *be the germ at* 0 *of a map into* $\emptyset[n+2, \mathbb{C}]$. *Then* $\gamma E\hat{\gamma}^{-1}$ *is holomorphic and invertible at* 0 *if and only if*

$$\hat{L}_+ = E(0)^{-1}L_+.$$

Proof. Write E as a power series:

$$E(\lambda) = \sum_{k \geq 0} \lambda^k E_k.$$

Comparing coefficients of λ, we see that $\gamma E\hat{\gamma}^{-1}$ is holomorphic at zero if and only if

1. $\pi_- E_0 \hat{\pi}_+ = 0$ (this is the coefficient of λ^{-2});
2. $\pi_0 E_0 \hat{\pi}_+ = \pi_- E_0 \hat{\pi}_0 = \pi_- E_1 \hat{\pi}_+ = 0$ (these are the components of the coefficient of λ^{-1}).

Now observe that

$$\pi_- E_0 \hat{\pi}_+ = \pi_0 E_0 \hat{\pi}_+ = 0$$

if and only if $E_0 \hat{L}_+ = L_+$ and then, since $E_0 \in \emptyset[n+2, \mathbb{C}]$,

$$L_+ \oplus L_0 = L_+^{\perp} = E_0(\hat{L}_+^{\perp}) = E_0(\hat{L}_+ \oplus \hat{L}_0)$$

whence $\pi_- E_0 \hat{\pi}_0$ vanishes automatically.

This leaves the term involving E_1. However, when $E_0 \hat{L}_+ = L_+$, we have $E_1 \hat{L}_+ = E_1 E_0^{-1} L_+$ and

$$E_1 E_0^{-1} = \left.\frac{\partial E}{\partial \lambda}\right|_{\lambda=0} \in \emptyset[n+2, \mathbb{C}]$$

so that $E_1 E_0^{-1}$ is skew-symmetric. Thus, since L is 1-dimensional[33], we have

$$(E_1 E_0^{-1} L_+, L_+) = 0$$

giving $\pi_- E_1 \hat{\pi}_+ = 0$.

Thus $\gamma E\hat{\gamma}^{-1}$ is holomorphic at zero if and only if $\hat{L}_+ = E(0)^{-1}L_+$. The invertibility now follows by applying this result to $\hat{\gamma}E^{-1}\gamma^{-1}$. □

Fix such a $\gamma = \lambda\pi_+ + \pi_0 + \lambda^{-1}\pi_-$ and set $g_- = \gamma \circ t_\alpha$:

$$g_-(\lambda) = \gamma\left(\frac{\alpha - \lambda}{\alpha + \lambda}\right),$$

with $\alpha^2 \in \mathbb{R}^\times$. Thus $g_- : \mathbb{P}^1 \setminus \{\pm\alpha\} \to \emptyset[n+2, \mathbb{C}]$ and $g_-(0) = 1$. We want $g_- \in \mathcal{G}_*^-$ which means imposing two further conditions: firstly, we must have

$$\tau g_-(\lambda) = g_-(-\lambda)$$

or, equivalently,

$$\tau\gamma(\lambda) = \gamma(1/\lambda).$$

In our setting, τ is conjugation by the reflection $\rho : \mathbb{C}^{n+2} \to \mathbb{C}^{n+2}$ in $\mathbb{R}^n = (\mathbb{R}^{1,1})^\perp$ so that this condition reads

$$\rho L_+ = L_-.$$

---

[33]It is at this point of the argument that we are really using the hypothesis that $\mathrm{Ad}\,\gamma$ has only simple poles.

In particular, this forces $\rho L_+ \neq L_+$ and shows that γ is completely determined by L_+ since $L_- = \rho L_+$ and $L_0 = (L_+ \oplus \rho L_+)^\perp$.

Secondly, we must impose the reality condition

$$\overline{g_-(\lambda)} = g_-(\bar{\lambda})$$

which amounts to

$$\overline{\gamma(\lambda)} = \begin{cases} \gamma(\bar{\lambda}) & \text{if } \alpha \in \mathbb{R}^\times; \\ \gamma(1/\bar{\lambda}) & \text{if } \alpha \in \sqrt{-1}\mathbb{R}^\times; \end{cases}$$

or, equivalently,

$$\overline{L_+} = \begin{cases} L_+ & \text{if } \alpha^2 > 0; \\ L_- & \text{if } \alpha^2 < 0. \end{cases}$$

Now we can put all this together: for $L \subset \mathbb{C}^{n+2}$ a 1-dimensional isotropic subspace with $\rho L \neq L$, let γ_L be the homomorphism $\mathbb{C}^\times \to \emptyset[n+2, \mathbb{C}]$ given by

$$\gamma_L(\lambda) = \lambda \pi_+ + \pi_0 + \lambda^{-1} \pi_-$$

with $\operatorname{Im} \pi_+ = L$, $\operatorname{Im} \pi_- = \rho L$ and $\operatorname{Im} \pi_0 = (L \oplus \rho L)^\perp$. Further, for $\alpha \in \mathbb{C}^\times$, set $p_{\alpha,L} = \gamma_L \circ t_\alpha$ so that

$$p_{\alpha,L}(\lambda) = \frac{\alpha - \lambda}{\alpha + \lambda} \pi_+ + \pi_0 + \frac{\alpha + \lambda}{\alpha - \lambda} \pi_-.$$

We have shown that the simple factors in $\mathcal{G}_*^-$ are precisely the $p_{\alpha,L}$ with either

1. $\alpha^2 > 0$ and $L = \ell^\mathbb{C}$, the complexification of $\ell \in \mathbb{P}(\mathrm{L})$ with $\rho \ell \neq \ell$, or,
2. $\alpha^2 < 0$ and L is the complexification of a light-line ℓ in $\mathbb{R}^n \oplus \sqrt{-1}\mathbb{R}^{1,1}$ with $\rho \ell \neq \ell$.

With all this in hand, we can now compute the dressing action of our simple factors. With an eye to proving Bianchi permutability, we formulate a slightly more general result:

Proposition 4.11. *Let $p_{\alpha,L} \in \mathcal{G}_*^-$ and let E be a germ at α of a holomorphic map into $\emptyset[n+2, \mathbb{C}]$ such that*

$$\overline{E(\lambda)} = E(\bar{\lambda}), \qquad \tau E(\lambda) = E(-\lambda).$$

Suppose further that $\rho(E(\alpha)^{-1}L) \neq E(\alpha)^{-1}L$. Then

1. *$p_{\alpha,E(\alpha)^{-1}L} \in \mathcal{G}_*^-$;*
2. *$p_{\alpha,L} E p_{\alpha,E(\alpha)^{-1}L}^{-1}$ is holomorphic and invertible at α.*

Proof. For the first assertion we must establish the reality condition for $p_{\alpha,E(\alpha)^{-1}L}$ and there are two cases. First, if $\alpha \in \mathbb{R}$, we must show that $\overline{E(\alpha)^{-1}L} = E(\alpha)^{-1}L$. However, in this case, $\overline{L} = L$ and $\overline{E(\alpha)} = E(\alpha)$ so this follows immediately.

When $\alpha \in \sqrt{-1}\mathbb{R}$, we must show that $\overline{E(\alpha)} = \rho E(\alpha)$ and, in this case, we have $\overline{L} = \rho L$ while

$$\overline{E(\alpha)} = E(\overline{\alpha}) = E(-\alpha) = \tau E(\alpha) = \rho \circ E(\alpha) \circ \rho^{-1}.$$

Thus

$$\overline{E(\alpha)^{-1}L} = \overline{E(\alpha)^{-1}} \rho L = \rho E(\alpha)^{-1} L$$

as required.

The second assertion follows at once from Lemma 4.10:

$$p_{\alpha,L} E p_{\alpha,E(\alpha)^{-1}L}^{-1} = (\gamma_L \circ t_\alpha) E (\gamma_{E(\alpha)^{-1}L} \circ t_\alpha)^{-1}$$

which is holomorphic at α if and only if

$$\gamma_L(E \circ t_\alpha^{-1})\gamma_{E(\alpha)^{-1}L}^{-1}$$

is holomorphic at 0. However, $E \circ t_\alpha^{-1}$ is holomorphic at 0 with value $E(\alpha)$ there so Lemma 4.10 applies. $\qquad\square$

As a corollary we have:

Theorem 4.12. $\mathcal{U}_{p_{\alpha,L}} = \{g_+ \in \mathcal{G}^+ : g_+(\alpha)^{-1}L \neq \langle v_0\rangle^\mathbb{C}, \langle v_\infty\rangle^\mathbb{C}\}$ *and, for* $g_+ \in \mathcal{U}_{p_{\alpha,L}}$,

$$(4.12) \qquad p_{\alpha,L}\# g_+ = p_{\alpha,L}g_+ p_{\alpha,g_+^{-1}(\alpha)L}^{-1}.$$

Proof. Let $g_+ \in \mathcal{G}^+$ be such that $g_+(\alpha)^{-1}L \neq \langle v_0\rangle^\mathbb{C}, \langle v_\infty\rangle^\mathbb{C}$. The first part of Proposition 4.11 assures us that $p_{\alpha,g_+^{-1}(\alpha)L} \in \mathcal{G}_*^-$ so all we need do is see that $p_{\alpha,L}\# g_+$ given by (4.12) defines an element of $\mathcal{G}^+$. It is clear that $p_{\alpha,L}\# g_+$ has the reality and twisting conditions as it is a product of maps with these conditions so the only issue is that of holomorphicity and invertibility at $\pm\alpha$. However, holomorphicity at α follows at once from Proposition 4.11 and then we get holomorphicity at $-\alpha$ from the twisting condition:

$$\tau(p_{\alpha,L}\# g_+)(\lambda) = p_{\alpha,L}\# g_+(-\lambda).$$

$\qquad\square$

Exercise 4.5. Complete the proof of Theorem 4.12 by showing that if $g_+ \in \mathcal{G}^+$ has $g_+^{-1}(\alpha)L = \langle v_0\rangle^\mathbb{C}$ or $\langle v_\infty\rangle^\mathbb{C}$ then $g_+ \notin \mathcal{U}_{p_{\alpha,L}}$.

4.5. **The action of simple factors on Christoffel pairs.** We are finally in a position to compute the dressing action of simple factors on Christoffel pairs of isothermic surfaces. Let us begin by recalling all the ingredients: a $\mathfrak{p}$-flat map $\psi : M \to \mathfrak{p}$ is the same as a Christoffel pair (f, f^c):

$$\psi = \begin{pmatrix} 0 & f \\ f^c & 0. \end{pmatrix}$$

$g_- \in \mathcal{G}_*^-$ acts on ψ by (4.9):

$$g_-\#\psi = \psi + \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0}\hat{g}_-^{-1}$$

where $\hat{g}_- : M \to \mathcal{G}_*^-$ comes from the factorisation

$$g_-\Phi = (g_-\#\Phi)\hat{g}_-$$

and $\Phi : M \to \mathcal{G}^+$ solves

$$\Phi^{-1}d\Phi = \lambda d\psi = \lambda \begin{pmatrix} 0 & df \\ df^c & 0 \end{pmatrix},$$
$$\Phi(o) = 1.$$

Now take $g_- = p_{\alpha,L}$. Then Theorem 4.12 gives

$$g_-\#\Phi = p_{\alpha,L}\Phi p_{\alpha,\Phi(\alpha)^{-1}L}^{-1}$$

so that $\hat{g}_- = p_{\alpha,\Phi(\alpha)^{-1}L}$ and we have

$$(4.13) \qquad p_{\alpha,L}\# \begin{pmatrix} 0 & f \\ f^c & 0 \end{pmatrix} = \begin{pmatrix} 0 & f \\ f^c & 0 \end{pmatrix} + \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} p_{\alpha,\Phi(\alpha)^{-1}L}^{-1}.$$

All that remains to do is to compute the second summand in (4.13). For this, write

$$\gamma_{\Phi^{-1}(\alpha)L}(\lambda) = \lambda\hat{\pi}_+ + \hat{\pi}_0 + \lambda^{-1}\hat{\pi}_-$$

so that $\operatorname{Im}\hat{\pi}_+ = \Phi^{-1}(\alpha)L$. Then

$$p_{\alpha,\Phi(\alpha)^{-1}L}(\lambda) = \frac{\alpha-\lambda}{\alpha+\lambda}\hat{\pi}_+ + \hat{\pi}_0 + \frac{\alpha+\lambda}{\alpha-\lambda}\hat{\pi}_-$$

so that

$$\left.\frac{\partial}{\partial\lambda}\right|_{\lambda=0} p_{\alpha,\Phi(\alpha)^{-1}L}^{-1} = \frac{2}{\alpha}(\hat{\pi}_+ - \hat{\pi}_-).$$

Lemma 4.13. *Fix* $\omega_o \in L^\times$ *and set* $\omega = \Phi^{-1}(\alpha)\omega_o : M \to \mathbb{C}^{n+2}$. *Then:*

1. ω *is the unique solution of*

(4.14a) $$\mathrm{d}\omega + \Phi_\alpha^{-1}\mathrm{d}\Phi_\alpha\omega = 0$$

(4.14b) $$\omega(o) = \omega_o.$$

2. *Viewing* $\emptyset$ *as* $[\mathbb{R}^{n+1,1}, \mathbb{R}^{n+1,1}] \subset C\ell_{n+1,1}$,

$$\hat{\pi}_+ - \hat{\pi}_- = \tfrac{1}{2}\frac{[\omega,\rho\omega]}{\{\omega,\rho\omega\}}.$$

Proof. We have $\omega_o = \Phi(\alpha)\omega$ and differentiating gives

$$0 = \mathrm{d}\Phi(\alpha)\omega + \Phi(\alpha)\mathrm{d}\omega$$

whence (4.14a). Further, $\Phi(o)(\alpha) = 1$ whence (4.14b).

For the second part, recall that under the isomorphism $[\mathbb{R}^{n+1,1}, \mathbb{R}^{n+1,1}] \cong \emptyset$, $\xi \in [\mathbb{R}^{n+1,1}, \mathbb{R}^{n+1,1}]$ acts on $\mathbb{R}^{n+1,1}$ by $v \mapsto [\xi, v]$. We must therefore show that, with $\xi = \tfrac{1}{2}[\omega,\rho\omega]/\{\omega,\rho\omega\}$, we have

$$[\xi, \omega] = \omega, \quad [\xi, \rho\omega] = -\rho\omega, \quad [\xi, v] = 0,$$

for $v \perp \langle\omega, \rho\omega\rangle$. For $v \perp \langle\omega, \rho\omega\rangle$, v anti-commutes with both ω and $\rho\omega$ and so commutes with $[\omega, \rho\omega]$. Again, using $\omega^2 = 0$, we have

$$[[\omega, \rho\omega], \omega] = (\omega\rho\omega - \rho\omega\omega)\omega - \omega(\omega\rho\omega - \rho\omega\omega)$$
$$= 2\omega\rho\omega\omega = 2\{\omega, \rho\omega\}\omega.$$

Similarly, we have

$$[[\omega, \rho\omega], \rho\omega] = -2\{\omega, \rho\omega\}\rho\omega.$$

$\square$

Write

$$\omega = \begin{pmatrix} v & s \\ t & -v \end{pmatrix}$$

so that $v \in \mathbb{C}^n$ and $s, t \in \mathbb{C}$ with $v^2 + st = 0$.

Exercise 4.6.

$$\hat{\pi}_+ - \hat{\pi}_- = \tfrac{1}{2}\begin{pmatrix} 0 & v/t \\ t/v & 0 \end{pmatrix}.$$

Thus, setting $h = v/t$, we have

$$p_{\alpha,L}\#\begin{pmatrix} 0 & f \\ f^c & 0 \end{pmatrix} = \begin{pmatrix} 0 & f + h/\alpha \\ f^c + h^{-1}/\alpha & 0 \end{pmatrix}$$

while (4.14a) reads

$$\mathrm{d}\begin{pmatrix} v & s \\ t & -v \end{pmatrix} + \left[\begin{pmatrix} 0 & \alpha\mathrm{d}f \\ \alpha\mathrm{d}f^c & 0 \end{pmatrix}, \begin{pmatrix} v & s \\ t & -v \end{pmatrix}\right] = 0.$$

We now argue as on page 34 to conclude that

(4.15) $\mathrm{d}h = \alpha h \mathrm{d}f^c h - \alpha \mathrm{d}f.$

Finally, set $g = h/\alpha = v/t\alpha$. Since g is homogeneous in the entries of ω, without loss of generality, we may take v to be $\mathbb{R}^n$-valued and $t \in \mathbb{R}$ or $\sqrt{-1}\mathbb{R}$ according to whether $\alpha \in \mathbb{R}$ or $\sqrt{-1}\mathbb{R}$. Either way, $t\alpha \in \mathbb{R}$ so that $g : M \to \mathbb{R}^n$ and (4.15) becomes the familiar Riccati equation

$$\mathrm{d}g = \alpha^2 g \mathrm{d}f^c g - \mathrm{d}f$$

while

$$p_{\alpha,L} \# \begin{pmatrix} 0 & f \\ f^c & 0 \end{pmatrix} = \begin{pmatrix} 0 & f + g \\ f^c + (\alpha^2 g)^{-1} & 0. \end{pmatrix}$$

Thus

$$p_{\alpha,L} \# \begin{pmatrix} 0 & f \\ f^c & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathcal{D}_{\alpha^2} f \\ \mathcal{D}_{\alpha^2} f^c & 0 \end{pmatrix}$$

and we have proved:

Theorem 4.14. *The dressing action of the simple factor $p_{\alpha,L}$ on a Christoffel pair (f, f^c) is by the Darboux transform $\mathcal{D}_{\alpha^2}^v$ where L is the complexification of the null-line corresponding to $\alpha(v - f(o))$.*

In particular, Darboux transforms $\mathcal{D}_r$ correspond to the two types of simple factor according to the sign of r.

Remark. Our action on $\mathfrak{p}$-flat maps is only local: $p_{\alpha,L} \# \Phi$ fails to be defined at points $p \in M$ where $\Phi(p) \notin \mathcal{U}_{p_{\alpha,L}}$, that is, when $\Phi^{-1}(p)(\alpha)L = \langle v_0 \rangle$ or $\langle v_\infty \rangle$. The geometric meaning of this restriction is now clear: these are the points where $g(p) = 0$ or $g(p) = \infty$ and so are exactly the singularities of our Riccati equation. In the first case, we have $f(p) = \hat{f}(p)$ and, in the second, $f^c(p) = \hat{f}^c(p)$. In either case, we have genuine singularities of the corresponding curved flats $(f, \hat{f})$ or $(f^c, \hat{f}^c)$.

4.6. Applications.
This new viewpoint on Darboux transformations allows several standard arguments from the loop group formalism to be applied. We conclude our study by considering some of these.

4.6.1. *Explicit solutions.*
In general, computation of a Darboux transform involves solving a differential equation: either the Riccati equation for g or, what is essentially the same thing, the Maurer–Cartan equations for the based extended frame Φ at $\lambda = \alpha$. However, the loop group approach has the following advantage: if one based extended frame is known then the based extended frame of any Darboux transform can be found *algebraically* via:

$$p_{\alpha,L} \# \Phi = p_{\alpha,L} \Phi p_{\alpha,\Phi(\alpha)^{-1}L}^{-1}.$$

In this way, one can iteratively construct infinitely many explicit examples given one known based extended frame—this is the procedure of "dressing the vacuum". The issue is, of course, to find a suitable "seed" Christoffel pair with known extended frame.

Experience with other problems (see, for example, [12]) suggests that a good starting point is to look for surfaces framed by a 2-dimensional abelian subgroup of G for then the Maurer–Cartan equations are solved by exponentiation and extended flat frames are readily computed.

For example: let e_1, e_2, e_3 denote the standard basis of $\mathbb{R}^3$ and let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be given by

$$f(x, y) = xe_1 + ye_2.$$

The plane parametrised by f is trivially isothermic with Christoffel transform

$$f^c(x, y) = xe_1 - ye_2$$

and the corresponding $\mathfrak{p}$-flat map has

$$\mathrm{d}\psi = E_1 \mathrm{d}x + E_2 \mathrm{d}y$$

where

$$E_1 = \begin{pmatrix} 0 & e_1 \\ e_1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 0 & e_2 \\ -e_2 & 0. \end{pmatrix}$$

Exercise 4.7. Show that $[E_1, E_2] = 0$, $E_1^2 = -1$ and $E_2^2 = 1$.

Thus the extended flat frame Φ based at $0 \in \mathbb{R}^2$ with $\Phi^{-1}\mathrm{d}\Phi = \lambda\mathrm{d}\psi$ is given by

$$\Phi(x, y)(\lambda) = (\exp \lambda x E_1)(\exp \lambda y E_2)$$
$$= \big(\cos \lambda x + (\sin \lambda x)E_1\big)\big(\cosh \lambda y + (\sinh \lambda y)E_1\big).$$

Having got our hands on Φ, we can compute the T-transforms of f:

$$T_r f = \Phi_{\sqrt{r}} \cdot 0.$$

Exercise 4.8. Show that

$$(\mathcal{T}_r f)(x, y) = \frac{(\sin 2\sqrt{r}x)e_1 + (\sinh 2\sqrt{r}y)e_2}{2(\cos^2 \sqrt{r}x + \sinh^2 \sqrt{r}y)}$$

Thus the T-transforms of f are different parametrisations of the same plane as is to be expected as all these isothermic surfaces share the same solution $\kappa \equiv 0$ of Calapso's equation.

Exercise 4.9. Compute the Darboux transforms of f.

A discussion of this example and its Darboux transforms can be found in [23].

A somewhat less trivial example arises as follows: set

$$E_3 = \begin{pmatrix} 0 & e_3 \\ e_3 & 0 \end{pmatrix}$$

$$E = E_{\mathfrak{k}} + E_{\mathfrak{p}} = \begin{pmatrix} e_1 e_2 & e_1 \\ -e_1 & e_1 e_2 \end{pmatrix}$$

and observe that $[E, E_3] = 0$. Taking $\mathfrak{k}$ and $\mathfrak{p}$ components gives (since $E_3 \in \mathfrak{p}$)

$$[E_{\mathfrak{k}}, E_3] = [E_{\mathfrak{p}}, E_3] = 0$$

so that

$$(E_{\mathfrak{k}} + \lambda E_{\mathfrak{p}})\mathrm{d}x + E_3\mathrm{d}y$$

solves the Maurer–Cartan equations for all λ and so integrates to give a frame $\hat{\Phi}$ of an associated family of curved flats. Indeed, since

$$E_3^2 = -1, \quad (E_{\mathfrak{k}} + \lambda E_{\mathfrak{p}})^2 = \lambda^2 - 1,$$

we readily compute that

$$\hat{\Phi}(x, y)(\lambda) = \exp\big(x(E_{\mathfrak{k}} + \lambda E_{\mathfrak{p}}) + \lambda y E_3\big)$$
$$= \Big(\cosh x\sqrt{\lambda^2 - 1} + \frac{\sinh x\sqrt{\lambda^2 - 1}}{\sqrt{\lambda^2 - 1}}(E_{\mathfrak{k}} + \lambda E_{\mathfrak{p}})\Big)\big(\cos \lambda y + (\sin \lambda y)E_3\big).$$

Now $\hat{\Phi}$ is not an extended flat frame since $\hat{\Phi}^{-1}\mathrm{d}\hat{\Phi}$ has non-zero $\mathfrak{k}$-component but the analysis of Section 3.1 assures us that gauging by $\hat{\Phi}_{|\lambda=0}$ gives such a frame. Thus

we define Φ by $\Phi = \hat{\Phi}\hat{\Phi}^{-1}_{|\lambda=0}$ to get a based extended flat frame with $\Phi(x,y)(\lambda)$ given by

$$\left(\cosh x\sqrt{\lambda^2-1} + \frac{\sinh x\sqrt{\lambda^2-1}}{\sqrt{\lambda^2-1}}(E_{\mathfrak{k}} + \lambda E_{\mathfrak{p}})\right)$$

$$\left(\cos \lambda y + (\sin \lambda y)E_3\right)\left(\cos x - (\sin x)E_{\mathfrak{k}}\right).$$

Exercise 4.10.   1. Show that
$$\left.\frac{\partial \Phi}{\partial \lambda}\right|_{\lambda=0} = \begin{pmatrix} 0 & f \\ f^c & 0 \end{pmatrix}$$

where

$$f(x,y) = \tfrac{1}{2}(\sin 2x)e_1 + \tfrac{1}{2}(1 - \cos 2x)e_2 + ye_3$$
$$f^c(x,y) = -\tfrac{1}{2}(\sin 2x)e_1 - \tfrac{1}{2}(1 - \cos 2x)e_2 + ye_3$$

so that the Christoffel pair associated to Φ is a right cylinder of radius $\tfrac{1}{2}$ (and so $H \equiv 1$) together with (up to a translation) the parallel (that is, identical) cylinder parametrised by $f + N$.
   2. Compute the T-transforms of the cylinder.
   3. Compute the Darboux transforms of the cylinder.
   4. Persuade a computer to draw pictures of the surfaces you have found.

A detailed analysis of this example and its Darboux transforms, using somewhat different methods, has been carried out by Bernstein [1].

4.6.2. *Bianchi permutability.* Recall the assertion of Theorem 2.19: given an isothermic surface f and Darboux transforms $f_i = \mathcal{D}_{r_i}f$, $i = 1, 2$, there is a fourth isothermic surface $\hat{f}$ such that
$$\hat{f} = \mathcal{D}_{r_1}f_2 = \mathcal{D}_{r_2}f_1.$$
Moreover, Theorem 2.20 says that the Christoffel transform of such a Bianchi quadrilateral is another such so that
$$\hat{f}^c = \mathcal{D}_{r_1}f_2^c = \mathcal{D}_{r_2}f_1^c.$$
In view of Theorem 4.14, both these results can be formulated in terms of simple factors: given a $\mathfrak{p}$-flat map ψ and Darboux transforms $\psi_1 = p_{\alpha_1, L_1}\#\psi$, $\psi_2 = p_{\alpha_2, L_2}\#\psi$, there is a $\mathfrak{p}$-flat map $\hat{\psi}$ and light-lines L_1', L_2' such that
$$\hat{\psi} = p_{\alpha_1, L_1'}\#(p_{\alpha_2, L_2}\#\psi) = p_{\alpha_2, L_2'}\#(p_{\alpha_1, L_1}\#\psi),$$
that is,
$$(p_{\alpha_1, L_1'}p_{\alpha_2, L_2})\#\psi = (p_{\alpha_2, L_2'}p_{\alpha_1, L_1})\#\psi.$$
We shall therefore have found an alternative (and simultaneous!) proof of both the Bianchi Permutability Theorem 2.19 and its Christoffel transform Theorem 2.20 as soon as we establish:

Proposition 4.15. *Let $p_{\alpha_i, L_i} \in \mathcal{G}_*^-$, $i = 1, 2$, with $\alpha_1^2 \neq \alpha_2^2$.*

Set

$$L_1' = p_{\alpha_2, L_2}(\alpha_1)L_1$$
$$L_2' = p_{\alpha_1, L_1}(\alpha_2)L_2$$

and assume that $L_i' \neq \langle v_0\rangle^{\mathbb{C}}, \langle v_\infty\rangle^{\mathbb{C}}$, $i = 1, 2$.

Then $p_{\alpha_1, L_i'} \in \mathcal{G}_*^-$, $i = 1, 2$ and

(4.16) $$p_{\alpha_1, L_1'}p_{\alpha_2, L_2} = p_{\alpha_2, L_2'}p_{\alpha_1, L_1}.$$

Proof. Since $\alpha_1 \neq \pm\alpha_2$, we have that p_{α_2, L_2}^{-1} is holomorphic near α_2 and so we may apply Proposition 4.11 with $E = p_{\alpha_2, L_2}^{-1}$ to conclude that $p_{\alpha_1, L_1'} \in \mathcal{G}_*^-$ and, further, that

$$p_{\alpha_1, L_1} p_{\alpha_2, L_2}^{-1} p_{\alpha_1, L_1'}^{-1}$$

is holomorphic and invertible at $\pm\alpha_1$.

Similarly $p_{\alpha_2, L_2'} \in \mathcal{G}_*^-$ and

$$p_{\alpha_2, L_2} p_{\alpha_1, L_1}^{-1} p_{\alpha_2, L_2'}^{-1}$$

is holomorphic and invertible at $\pm\alpha_2$.

Now contemplate

$$p_{\alpha_1, L_1'}(p_{\alpha_2, L_2} p_{\alpha_1, L_1}^{-1} p_{\alpha_2, L_2'}^{-1}) = (p_{\alpha_1, L_1} p_{\alpha_2, L_2}^{-1} p_{\alpha_1, L_1'}^{-1})^{-1} p_{\alpha_2, L_2'}^{-1}.$$

Looking at the left hand side, we see that this expression is holomorphic at $\pm\alpha_2$ and, from the right hand side, we see that is is holomorphic at $\pm\alpha_1$. Thus it is holomorphic on $\mathbb{P}^1$ and so constant. Evaluating at $\lambda = 0$ now gives

$$p_{\alpha_1, L_1'}(p_{\alpha_2, L_2} p_{\alpha_1, L_1}^{-1} p_{\alpha_2, L_2'}^{-1}) = 1$$

that is

$$p_{\alpha_1, L_1'} p_{\alpha_2, L_2} = p_{\alpha_2, L_2'} p_{\alpha_1, L_1}.$$

$\square$

There is another way to think about this result which shows what a general phenomenon it is that we are dealing with here: (4.16) amounts to a factorisation

$$p_{\alpha_1, L_1} p_{\alpha_2, L_2}^{-1} = p_{\alpha_2, L_2'}^{-1} p_{\alpha_1, L_1'}$$

corresponding to the subgroups $\mathcal{G}_{\alpha_i}$ of $\mathcal{G}_*^-$ consisting of those $g \in \mathcal{G}_*^-$ that are holomorphic on $\mathbb{P}^1 \setminus \{\pm\alpha_i\}$. Just as before, we get from such a factorisation a local action of $\mathcal{G}_{\alpha_1}$ on $\mathcal{G}_{\alpha_2}$ which we denote by $*_{\alpha_1}$ and then

$$p_{\alpha_2, L_2'} = (p_{\alpha_1, L_1} *_{\alpha_1} p_{\alpha_2, L_2}^{-1})^{-1}.$$

More generally, for $g_i \in \mathcal{G}_{\alpha_i}$, we can find $g_i' \in \mathcal{G}_{\alpha_i}$ with

$$g_1' g_2 = g_2' g_1$$

by setting

$$g_2' = (g_1 *_{\alpha_1} g_2^{-1})^{-1}, \quad g_1' = (g_2 *_{\alpha_2} g_1^{-1})^{-1}.$$

This shows that Bianchi permutability is not a consequence of the fact that our simple factors have simple poles but rather that these factors have only two poles.

In auspicious circumstances (for example $\alpha \in \sqrt{-1}\mathbb{R}$ and G compact) one can argue as in Section 4.3 and precompose everything with $t_{\alpha_2}^{-1}$ to reduce $*_{\alpha_1}$ to the globally defined Pressley–Segal action. For example, with $G = \mathrm{SU}(2)$, this accounts for the classical Bäcklund transform of pseudo-spherical surfaces and their Bianchi permutability.

As a final advertisement for this technology, let us give another proof of Theorem 2.22 which asserts that the Darboux transform of a Bianchi quadrilateral is another Bianchi quadrilateral thus giving a configuration of 8 isothermic surfaces forming the vertices of a cube all of whose faces are Bianchi quadrilaterals. For this, choose $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{C}^\times$ with all α_i^2 real and distinct and let $q_i \in \mathcal{G}_{\alpha_i}$ be three

simple factors with poles at $\pm\alpha_i$. Proposition 4.15 now gives simple factors $q_i^j \in \mathcal{G}_{\alpha_i}$ with

(4.17a) $$q_3^1 q_1 = q_1^3 q_3$$

(4.17b) $$q_1^2 q_2 = q_2^1 q_1$$

(4.17c) $$q_2^3 q_3 = q_3^2 q_2$$

and then simple factors $q_i^{jj} \in \mathcal{G}_{\alpha_i}$ with

(4.18a) $$q_1^{33} q_2^3 = q_2^{33} q_1^3$$

(4.18b) $$q_1^{22} q_3^2 = q_3^{22} q_1^2$$

(4.18c) $$q_2^{11} q_3^1 = q_3^{11} q_2^1.$$

(This notation becomes a little easier to stomach when one sees that the subscripts locate the poles of the simple factor. Figure 3 on page 71 may also help.)

The key to our result is the following lemma that asserts that the q_i^{jj} are determined solely by their poles:

Lemma 4.16. $q_1^{22} = q_1^{33}$, $q_2^{11} = q_2^{33}$, $q_3^{11} = q_3^{22}$.

Proof. Multiply (4.18a) by q_3 to get

$$q_1^{33} q_2^3 q_3 = q_2^{33} q_1^3 q_3$$

and use (4.17a) and (4.17c) to get

$$q_1^{33} q_3^2 q_2 = q_2^{33} q_3^1 q_1.$$

Rearranging this and using (4.17b) yields

$$q_1^{33} q_3^2 = q_2^{33} q_3^1 q_1 q_2^{-1} = q_2^{33} q_3^1 (q_2^1)^{-1} q_1^2$$

whence

(4.19) $$q_1^{33} q_3^2 (q_1^2)^{-1} = q_2^{33} q_3^1 (q_2^1)^{-1}.$$

Temporarily denote by q the common value in (4.19). From the left hand side, we see that q is holomorphic except possibly at $\pm\alpha_1, \pm\alpha_3$ while the right hand side tells us that q is holomorphic except possibly at $\pm\alpha_2, \pm\alpha_3$. We therefore conclude that q has poles at $\pm\alpha_3$ only, that is, $q \in \mathcal{G}_{\alpha_3}$ so that we have factorisations

$$q_1^{33} q_3^2 = q q_1^2$$
$$q_2^{33} q_3^1 = q q_2^1.$$

However, for $i \neq j$, $\mathcal{G}_{\alpha_i} \cap \mathcal{G}_{\alpha_j} = \{1\}$ so factorisations of this kind are unique (recall Exercise 4.1!) and, comparing with (4.18b), (4.18c), we get

$$q_1^{33} = q_1^{22} \qquad\qquad q = q_3^{22}$$
$$q_2^{33} = q_2^{11} \qquad\qquad q = q_3^{11}.$$

$\square$

With this in hand, start with a $\mathfrak{p}$-flat map ψ and set

$$\psi_1 = q_1 \# \psi, \quad \psi_2 = q_2 \# \psi, \quad \psi' = q_3 \# \psi.$$

We then obtain Bianchi quadrilaterals $(\psi, \psi_1, \hat{\psi}, \psi_2)$, $(\psi, \psi', \psi_1', \psi_1)$, $(\psi, \psi', \psi_2', \psi_2)$ with

$$\hat{\psi} = (q_1^2 q_2)\#\psi = (q_2^1 q_1)\#\psi$$
$$\psi_1' = (q_1^3 q_3)\#\psi = (q_3^1 q_1)\#\psi$$
$$\psi_2' = (q_2^3 q_3)\#\psi = (q_3^2 q_2)\#\psi$$

and then a Bianchi quadrilateral $(\psi', \psi_1', \hat{\psi}', \psi_2')$ with

$$\hat{\psi}' = (q_1^{33} q_2^3)\#\psi' = (q_2^{33} q_1^3)\#\psi'.$$

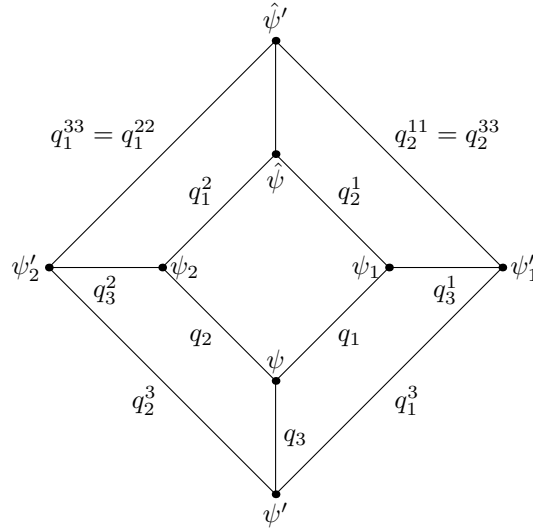The situation is summarised in Figure 3. The claim is that the remaining two faces

$$\hat{\psi}'$$

$$q_1^{33} = q_1^{22} \qquad q_2^{11} = q_2^{33}$$

$$q_1^2 \quad \hat{\psi} \quad q_2^1$$

$$\psi_2' \qquad \psi_2 \qquad \psi_1 \qquad \psi_1'$$

$$q_3^2 \qquad \qquad q_3^1$$

$$q_2 \qquad q_1$$

$$\psi$$

$$q_2^3 \qquad \qquad q_1^3$$

$$q_3$$

$$\psi'$$

FIGURE 3. Darboux transform of a Bianchi quadrilateral

$(\psi_2, \hat{\psi}, \hat{\psi}', \psi_2')$ and $(\psi_1, \psi_1'\hat{\psi}', \hat{\psi})$ are also Bianchi quadrilaterals. That is,

$$\hat{\psi}' = (q_1^{22} q_3^2)\#\psi_2 = (q_2^{11} q_3^1)\#\psi_1.$$

But Lemma 4.16 with (4.17c) gives

$$(q_1^{22} q_3^2)\#\psi_2 = (q_1^{33} q_3^2 q_2)\#\psi$$
$$= (q_1^{33} q_2^3 q_3)\#\psi = (q_1^{33} q_2^3)\#\psi' = \hat{\psi}'.$$

A similar argument establishes the second equation.

While this argument requires some book-keeping it seems less involved than our Clifford algebra cross-ratio argument of Section 2.4 and has a certain universal character which applies to all other Bäcklund transforms which are given by the dressing action of simple factors. For example, working with $G = \mathrm{SU}(2)$ and the extended frames of pseudo-spherical surfaces, we immediately read off a result which was doubtless known to Bianchi:

Theorem 4.17. *The Bäcklund transform of a Bianchi quadrilateral of pseudo-spherical surfaces is another such.*

5. Coda

We have developed a fairly complete theory of isothermic surfaces in $\mathbb{R}^n$ but there is more to be said and more to be understood. I draw this (already over-long) work to a close by indicating some recent developments in the area and some open problems.

5.1. **Recent developments.**

5.1.1. *Symmetric R-spaces.* The conformal geometry of S^n is an example of a parabolic geometry of a kind possessed by any *symmetric R-space* [49, 50, 64]. According to Nagano [55], these can be characterised as those Riemannian symmetric spaces of compact type which admit a Lie groups of diffeomorphisms strictly larger than the isometry group. Thus examples include:

1. S^n with its group Möb(n) of conformal diffeomorphisms and more generally the conformal compactification $S^p \times S^q$ of $\mathbb{R}^{p,q}$ with the corresponding group of conformal diffeomorphisms;
2. Any Grassmannian $G_k(\mathbb{R}^n)$ of k-planes in $\mathbb{R}^n$ with the action of PSL$(n, \mathbb{R})$. In particular, taking $k = 1$, we find the setting of projective differential geometry.
3. Any Hermitian symmetric space of compact type with its group of biholomorphisms.

All symmetric R-spaces have a common algebraic structure[34] which accounts for all the structure we have exploited in this work: one has analogues of stereographic projection, the pseudo-Riemannian symmetric space Z of point pairs and, most importantly, an invariant formulation of the notion of an isothermic submanifold. Christoffel, Darboux and T-transformations are all available in this general context and the delicate inter-relations between them remain true as does the loop group interpretation described in section 4.

In particular, these ideas provide a manifestly conformally invariant definition of an isothermic surface in S^n, the lack of which may be viewed as a weakness of the present work.

These ideas will be described in [14].

5.1.2. *Meromorphic functions as isothermic surfaces.* One can specialise our existing theory to the case $n = 2$: this amounts to studying meromorphic functions on a Riemann surface M. In this case, the isothermic surface condition is vacuous—any meromorphic function is isothermic—so one must change one's point of view and emphasis the role of the holomorphic quadratic differential Q. Thus, on a polarised Riemann surface (M, Q), the Christoffel transform f^c of a meromorphic function f is given by specialising (2.3) to this setting and demanding

$$\partial f^c = Q/\partial f.$$

If f is viewed as the Gauss map of a minimal surface with Hopf differential Q via the Weierstrass–Enneper formula, this transformation gives rise to an intriguing transformation of minimal surfaces that has been studied by McCune [54].

---

[34]The stabilisers of points in the "big" group are parabolic subgroups with *abelian* nilradical.

5.1.3. *Willmore surfaces in S^4.* In low dimensions, Clifford algebras are most conveniently studied as transformations of spinors. For $n = 4$, this amounts to viewing S^4 as the quaternionic projective line $\mathbb{H}P^1$. Here a central topic is the study of Willmore surfaces—extremals of a conformally invariant functional that are characterised by the harmonicity of their conformal Gauss map. There are strong formal analogies between such conformal Gauss maps and the Euclidean Gauss maps of CMC surfaces. One can exploit this analogy along with the methods of Section 2.3 to obtain a large family of "Darboux" transformations of Willmore surfaces.

Similarly, another class of transformations can be obtained by adapting the methods of McCune [54] to this context.

A detailed exposition of these ideas may be found in [8].

5.2. **Open problems.** I list some problems to which I would like to know the answers!

1. Is there any interesting theory of isothermic submanifolds of $\mathbb{R}^n$ of dimension greater than two? The problem here is to find a suitable definition that is not too restrictive: certainly our formulation only works in 2 dimensions and the same is true of the symmetric R-space approach. One way forward might be to study submanifolds admitting a conformal Ribaucour sphere congruence. The work of Dajczer–Tojeiro [25] may be relevant here.
2. Motivated by considerations concerning surfaces isometric to quadrics that this writer does not understand, Darboux [28] distinguished the class of *special isothermic surfaces* in $\mathbb{R}^3$ and these were studied intensively by Bianchi [2, 3] and Calapso [16]. Characterised by a differential equation on the mean curvature, this class includes CMC surfaces as a degenerate case and is stable under all the transformations of the theory.

   Problem. *Find a simple geometric characterisation of special isothermic surfaces in $\mathbb{R}^3$.*
      Is there an interesting extension of the notion to surfaces in $\mathbb{R}^n$?

3. The theory of constant mean curvature (CMC) surfaces in $\mathbb{R}^3$ lies at the intersection of two integrable geometries: via their Gauss maps, they are the same as harmonic maps into S^2, a well-studied integrable system. In particular, they admit a spectral deformation, the "associated family", through CMC surfaces f_μ for $\mu \in S^1$. On the other hand, viewed as isothermic surfaces, they have the spectral deformation $\mathcal{T}_r f$ through isothermic surfaces for $r \in \mathbb{R}$ which amounts to the Guichard–Lawson deformation through CMC surfaces in other space forms (see [43] for a recent account). The relation between these deformations is not well-understood although there is some evidence to suggest that they should be viewed as the angular and radial parts of a single complex deformation[35].
      Again, the Darboux transforms of CMC surfaces described herein amount to the (iterated) Bäcklund transforms of the harmonic map theory [44] despite the fact that the underlying symmetry groups seem quite different. Thus we formulate:

   Problem. *Find a theory of CMC surfaces that unifies the harmonic map and isothermic surface theories.*

---

[35]*Note added in December 2001*: this issue has now been clarified in [13]

References

[1] H. Bernstein, *Non-special, non-canal, isothermic tori with spherical lines of curvature*, Trans. Amer. Math. Soc. **353** (2001), 2245–2274 (electronic).

[2] L. Bianchi, *Ricerche sulle superficie isoterme e sulla deformazione delle quadriche*, Ann. di Mat. **11** (1905), 93–157.

[3] ———, *Complementi alle ricerche sulle superficie isoterme*, Ann. di Mat. **12** (1905), 19–54.

[4] A.I. Bobenko and U. Pinkall, *Discrete isothermic surfaces*, J. reine angew. Math. **475** (1996), 187–208.

[5] A.I. Bobenko and W.K. Schief, *Discrete Indefinite Affine Spheres*, Discrete integrable geometry and physics (A.I. Bobenko and R. Seiler, eds.), Oxford University Press, Oxford, 1999, pp. 113–138.

[6] M. Brück, X. Du, J.S. Park, and C.-L. Terng, *The submanifold geometries associated to Grassmannian systems*, Mem. Amer. Math. Soc. **155** (2002), no. 735.

[7] R.L. Bryant, *A duality theorem for Willmore surfaces*, J. Diff. Geom. **20** (1984), 23–53.

[8] F. Burstall, D. Ferus, K. Leschke, F. Pedit, and U. Pinkall, *Conformal geometry of surfaces in the 4-sphere and quaternions*, Lect. Notes Math., vol. 1772, Springer Verlag, Heidelberg, Berlin, New York, 2001.

[9] F.E. Burstall, D. Ferus, F. Pedit, and U. Pinkall, *Harmonic tori in symmetric spaces and commuting Hamiltonian systems on loop algebras*, Ann. of Math. **138** (1993), 173–212.

[10] F.E. Burstall and M.A. Guest, *Harmonic two-spheres in compact symmetric spaces, revisited*, Math. Ann. **309** (1997), 541–572.

[11] F.E. Burstall, U. Hertrich-Jeromin, F. Pedit, and U. Pinkall, *Curved flats and isothermic surfaces*, Math. Z. **225** (1997), 199–209.

[12] F.E. Burstall and F. Pedit, *Dressing orbits of harmonic maps*, Duke Math. J. **80** (1995), 353–382.

[13] F.E. Burstall, F. Pedit, and U. Pinkall, *Schwarzian Derivatives and Flows of Surfaces*, Differential geometry and integrable systems (Tokyo, 2000), Contemp. Math., vol. 308, Amer. Math. Soc., Providence, RI, 2002, pp. 39–61.

[14] ———, *Isothermic submanifolds of symmetric R-spaces*, in preparation.

[15] P. Calapso, *Sulle superficie a linee di curvatura isoterme*, Rendiconti Circolo Matematico di Palermo **17** (1903), 275–286.

[16] ———, *Sulle trasformazioni delle superficie isoterme*, Ann. di Mat. **24** (1915), 11–48.

[17] D.M.J. Calderbank, *Möbius structures and two dimensional Einstein–Weyl geometry*, J. reine angew. Math. **504** (1998), 37–53.

[18] A. Cayley, *On the surfaces divisible into Squares by their Curves of Curvature*, Proc. London Math. Soc. **4** (1872), 8–9, 120–121.

[19] T.E. Cecil, *Lie sphere geometry*, Springer-Verlag, New York, 1992.

[20] B. Chen, *Surfaces with a parallel isoperimetric section*, Bull. Amer. Math. Soc. **79** (1973), 599–600.

[21] E. Christoffel, *Ueber einige allgemeine Eigenshaften der Minimumsflächen*, Crelle's J. **67** (1867), 218–228.

[22] J. Cieśliński, *The cross ratio and Clifford algebras*, Adv. Appl. Clifford Algebras **7** (1997), 133–140.

[23] J. Cieśliński, *The Darboux-Bianchi transformation for isothermic surfaces. Classical results versus the soliton approach*, Differential Geom. Appl. **7** (1997), 1–28.

[24] J. Cieśliński, P. Goldstein, and A. Sym, *Isothermic surfaces in E^3 as soliton surfaces*, Phys. Lett. A **205** (1995), 37–43.

[25] M. Dajczer and R. Tojeiro, *Commuting Codazzi tensors and the Ribaucour transformation for submanifolds*, Results Math. **44** (2003), 258–278.

[26] G. Darboux, *Leçons sur la théorie générale des surfaces et les applications géomtriques du calcul infinitésimal. Parts 1 and 2*, Gauthier-Villars, Paris, 1887.

[27] ———, *Sur les surfaces isothermiques*, C.R. Acad. Sci. Paris **128** (1899), 1299–1305.

[28] ———, *Sur une classe de surfaces isothermiques liées à la déformations des surfaces du second degré*, C.R. Acad. Sci. Paris **128** (1899), 1483–1487.

[29] M.P. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall Inc., Englewood Cliffs, N.J., 1976.

[30] P.G. Drazin and R.S. Johnson, *Solitons: an introduction*, Cambridge University Press, Cambridge, 1989.

[31] B. Dubrovin, *Integrable systems and classification of 2-dimensional topological field theories*, Integrable systems (Luminy, 1991), Birkhäuser Boston, Boston, MA, 1993, pp. 313–359.

[32] L.P. Eisenhart, *A treatise on the differential geometry of curves and surfaces*, Dover Publications Inc., New York, 1960.

[33] N. Ejiri, *Willmore surfaces with a duality in $S^N(1)$*, Proc. London Math. Soc. (3) **57** (1988), 383–416.

[34] J. Elstrodt, F. Grunewald, and J. Mennicke, *Vahlen's group of Clifford matrices and spin-groups*, Math. Z. **196** (1987), 369–390.

[35] ———, *Kloosterman sums for Clifford algebras and a lower bound for the positive eigen-values of the Laplacian for congruence subgroups acting on hyperbolic spaces*, Invent. Math. **101** (1990), 641–685.

[36] L.D. Faddeev and L.A. Takhtajan, *Hamiltonian methods in the theory of solitons*, Springer-Verlag, New York–Heidelberg–Berlin, 1987.

[37] E. V. Ferapontov, *Surfaces in Lie sphere geometry and the stationary Davey-Stewartson hierarchy*, preprint dg-ga/9710028v1, 1997.

[38] D. Ferus and F. Pedit, *Curved flats in symmetric spaces*, Manuscripta Math. **91** (1996), 445–454.

[39] ———, *Isometric immersions of space forms and soliton theory*, Math. Ann. **305** (1996), 329–342.

[40] J.E. Gilbert and M.A.M. Murray, *Clifford algebras and Dirac operators in harmonic analysis*, Cambridge University Press, Cambridge, 1991.

[41] U. Hertrich-Jeromin, *Supplement on curved flats in the space of point pairs and isothermic surfaces: a quaternionic calculus*, Doc. Math. **2** (1997), 335–350.

[42] U. Hertrich-Jeromin, T. Hoffmann, and U. Pinkall, *A discrete version of the Darboux trans-form for isothermic surfaces*, Discrete integrable geometry and physics (A. Bobenko and R. Seiler, eds.), OUP, Oxford, 1999.

[43] U. Hertrich-Jeromin, E. Musso, and L. Nicolodi, *Moebius geometry of surfaces of constant mean curvature 1 in hyperbolic space*, Ann. Global Anal. Geom. **19** (2001), 185–205.

[44] U. Hertrich-Jeromin and F. Pedit, *Remarks on the Darboux transform of isothermic surfaces*, Doc. Math. **2** (1997), 313–333.

[45] N. Hitchin, *Frobenius manifolds (notes by D. Calderbank)*, Gauge theory and symplectic geometry (Montreal, PQ, 1995), Kluwer Acad. Publ., Dordrecht, 1997, pp. 69–112.

[46] N.J. Hitchin, *Harmonic maps from a 2-torus to the 3-sphere*, J. Diff. Geom. **31** (1990), 627–710.

[47] G. Kamberov, *Quadratic Differentials, Quaternionic Forms, and Surfaces*, preprint dg-ga/9712011, 1997.

[48] G. Kamberov, F. Pedit, and U. Pinkall, *Bonnet pairs and isothermic surfaces*, Duke Math. J. **92** (1998), 637–644.

[49] S. Kobayashi and T. Nagano, *On filtered Lie algebras and geometric structures. I*, J. Math. Mech. **13** (1964), 875–907.

[50] ———, *On filtered Lie algebras and geometric structures. II*, J. Math. Mech. **14** (1965), 513–521.

[51] B. G. Konopelchenko, *Weierstrass representations for surfaces in 4D spaces and their inte-grable deformations via DS hierarchy*, Ann. Glob. Anal. Geom. **18** (2000), 61–74.

[52] B.G. Konopelchenko and G. Landolfi, *Induced surfaces and their integrable dynamics. II. Generalized Weierstrass representations in 4D spaces and deformations via DS hierarchy*, Stud. Appl. Math. **104** (2000), 129–169.

[53] H.B. Lawson and M. Michelsohn, *Spin geometry*, Princeton University Press, Princeton, NJ, 1989.

[54] C. McCune, *Rational minimal surfaces*, Quart. J. Math. **52** (2001), 329–354.

[55] T. Nagano, *Transformation groups on compact symmetric spaces*, Trans. Amer. Math. Soc. **118** (1965), 428–453.

[56] B. Palmer, *Isothermic surfaces and the Gauss map*, Proc. Amer. Math. Soc. **104** (1988), 876–884.

[57] F. Pedit and U. Pinkall, *Quaternionic analysis on Riemann surfaces and differential geom-etry*, ICM 1998 proceedings, volume 2, 1998, pp. 389–400.

[58] I.R. Porteous, *Clifford Algebras and the Classical Groups*, Cambridge Univ. Press, Cam-bridge, 1995.

[59] A.N. Pressley and G. Segal, *Loop Groups*, Oxford Math. Monographs, Clarendon Press, Oxford, 1986.

[60] R. Rothe, *Untersuchungen über die Theorie der isothermen Flächen*, Berlin thesis, 1897.

[61] B.A. Rozenfel'd, *Conformal differential geometry of families of C_m in C_n*, Mat. Sbornik N.S. **23(65)** (1948), 297–313.

[62] W.K. Schief, *Isothermic surfaces in spaces of arbitrary dimension: Integrability, discretiza-tion and Bäcklund transformations. A discrete Calapso equation*, Stud. Appl. Math., **106** (2001), 85–137.

[63] S. Sternberg, *Lectures on differential geometry*, Chelsea, New York, 1983.

[64] M. Takeuchi, *Cell decompositions and Morse equalities on certain symmetric spaces*, J. Fac. Sci. Univ. Tokyo Sect. I **12** (1965), 81–192.

[65] C.-L. Terng, *Soliton equations and differential geometry*, J. Diff. Geom. **45** (1997), 407–445.

[66] C.-L. Terng and K. Uhlenbeck, *Poisson actions and scattering theory for integrable systems*, Integrable systems (C.-L. Terng and K. Uhlenbeck, eds.), Surveys in Differential Geometry, vol. 4, International Press, Cambridge MA, 1998, pp. 315–402.

[67] _____, *Bäcklund transformations and loop group actions*, Comm. Pure Appl. Math. **53** (2000), 1–75.

[68] G. Thomsen, *Ueber konforme Geometrie I: Grundlagen der konformen Flaechentheorie*, Hamb. Math. Abh. **3** (1923), 31–56.

[69] G. Tzitzéica, *Géométrie différentielle projective des réseaux*, Cultura Nationala, Bucharest, 1924.

[70] K. Uhlenbeck, *Harmonic maps into Lie groups (classical solutions of the chiral model)*, J. Diff. Geom. **30** (1989), 1–50.

[71] _____, *On the connection between harmonic maps and the self-dual Yang-Mills and the sine-Gordon equations*, J. Geom. Phys. **8** (1992), 283–316.

[72] K.T. Vahlen, *Ueber Bewegungen und complexe Zahlen*, Math. Ann. **55** (1902), 585–593.

[73] M. Wada, *Conjugacy invariants of Möbius transformations*, Complex Variables Theory Appl. **15** (1990), 125–133.

[74] V.E. Zakharov and A.V. Mikhaĭlov, *Relativistically invariant two-dimensional models of field theory which are integrable by means of the inverse scattering problem method*, Zh. Èksper. Teoret. Fiz. **74** (1978), 1953–1973.

[75] V.E. Zakharov and A.B. Shabat, *Integration of non-linear equations of mathematical physics by the method of inverse scattering, II*, Funkts. Anal. i Prilozhen **13** (1978), 13–22.

Department of Mathematical Sciences, University of Bath, Bath BA2 7AY, UK

E-mail address: feb@maths.bath.ac.uk